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ABSTRACT

This dissertation reports the results of a one-year pilot study conducted to determine the feasibility of a small group-discovery method with a class of college freshmen studying calculus. The class consisted of twelve volunteer students who had A or B grades in their high school mathematics courses. The pilot class scored slightly better on a final exam than a control section taught by the lecture-discussion system. The pilot class also had a positive or non-negative effect upon students' interest in mathematics and estimate of problem solving skill. A number of problems were stated for further investigation. Contents include: background, the small group-discovery method, classroom organization and social climate, the interaction between the students and the mathematics content, and data analysis. (MP)

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Technical Report No. 168

THE SMALL GROUP-DISCOVERY METHOD OF MATHEMATICS INSTRUCTION AS APPLIED IN CALCULUS

Report from Phase 2 of the Project on Prototypic Instructional Systems in Elementary Mathematics

By Neil Andrew Davidson

Thomas A. Romberg and John G. Harvey
Principal Investigators

Wisconsin Research and Development
Center for Cognitive Learning
The University of Wisconsin
Madison, Wisconsin

October 1971

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This Technical Report is from Phase 2 of the Project on Prototypic Instructional Systems in Elementary Mathematics in Program 2. General objectives of the Program are to establish rationale and strategy for developing instructional systems, to identify sequences of concepts and cognitive skills, to develop assessment procedures for those concepts and skills, to identify or develop instructional materials associated with the concepts and cognitive skills, and to generate new knowledge about instructional procedures. Contributing to the Program objectives, the Mathematics Project, Phase 1, is developing and testing a televised course in arithmetic for Grades 1-6 which provides not only a complete program of instruction for the pupils but also inservice training for teachers. Phase 2 has a long-term goal of providing an individually guided instructional program in elementary mathematics. Preliminary activities include identifying instructional objectives, student activities, teacher activities materials, and assessment procedures for integration into a total mathematics curriculum. The third phase focuses on the development of a computer system for managing individually guided instruction in mathematics and on a later extension of the system's applicability.

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ABSTRACT

This paper reports the development of a new strategy of mathematics instruction in which students work together in small groups to discover mathematical results. The strategy is named the "small group-discovery method." To determine the feasibility of the method, a one-year pilot study was conducted with a class of college freshmen studying calculus.

The strategy was designed in accordance with the educational philosophy of John Dewey. The students learned mathematics by doing mathematics, and the approach to the subject matter was one of guided discovery. The students, with some guidance from the teacher, stated and proved the theorems of calculus and developed techniques for solving various classes of problems. The classroom activity was a social process taking place in small groups of students, with four members per group.

The teacher spent most of the period with the small work groups— giving hints, checking solutions, providing encouragement, and so forth. The teacher provided mathematical input in the form of questions for investigation with appropriate terminology; this was done in brief discussions with the entire class, generally lasting only five or ten minutes per day. Instead of a textbook for the course, there was a set of dittoed notes prepared by the teacher.

Interest in the mathematics was to provide the major source of motivation, and there was a non-threatening atmosphere in which students could freely express their opinions. Emphasis upon grades was substantially reduced by an A-B grading scale, and there were no examinations in class. The teacher attempted to determine which topics had intrinsic value, which appeared to be useful, and which had no value from the student viewpoint. Professional standards of rigor were not imposed upon the beginner, and the presentation of subject matter proceeded from the more concrete to the more abstract. Skills were formed under conditions requiring thought, and there were opportunities for the exploration of student-generated questions.

Decisions about the classroom social climate were made on the basis of experimental studies in group dynamics. The teacher made use of a carefully specified style of democratic leadership, rather than an autocratic or laissezfaire style. Within each work group there was to be an atmosphere of cooperation, rather than competition. Conformity pressure

was reduced by developing standards which encouraged members to conform to their own judgment, and the work groups operated without designated leaders. The teacher exerted his influence to limit the discrepancy in power between the most active and the least active group members, and to facilitate the participation of all members in the discussion. The total class consisted of twelve members, who worked in three small groups.

The students in the one-year pilot class were all volunteers who were at least mildly interested in mathematics and who had A or B grades in their high school mathematics courses. On a comparative final examination, the students in the pilot class scored slightly better, but not significantly better, than the students in a control section taught by the lecture-discussion system. However, some students in the control section did not satisfy the entrance requirements for the discovery class.

The students in the discovery class performed very credibly on seven take-home examinations. An open-ended questionnaire showed that the pilot class had either positive or non-negative effects upon each student's interest in mathematics and estimate of his problem solving skill. The study indicated that the small group-discovery method deserves intensive investigation as an approach for instruction and curriculum development, and a number of problems were stated for further investigation.

John G. Harvey
Associate Professor

Chapter I

BACKGROUND

1.1 A BRIEF DESCRIPTION OF THE PROBLEM AND ITS SOLUTION

This paper reports the development of a new strategy of mathematics instruction in which students work together in small groups to discover mathematical results. The strategy is named the "small group-discovery method" of mathematics instruction.

A one-year pilot study was conducted with a class of college freshmen, and the first-year calculus was the subject matter used as a vehicle in the development of the instructional strategy. It was shown that an entire first-year calculus course could be taught by the small group-discovery method, and a limited, formative evaluation was conducted to measure the results of the approach. Certain innovations were made in the presentation of the calculus content in order to demonstrate that the small group-discovery method could be used for curriculum development in mathematics.

1.2 CHAPTER OUTLINE OF THE PAPER

Chapter I presents a critical review of the literature pertaining to existing methods of calculus instruction. The critical appraisal leads to a proposal for the development of a new strategy of mathematics instruction which combines small group methods with discovery learning.

Chapter II describes a systematic philosophy of education--John Dewey's educational philosophy of Experimentalism. The philosophy is then used to design the small group-discovery method of mathematics instruction.

Chapter III is an elaboration of the design in the previous chapter, placing emphasis upon the classroom social climate. A summary of pertinent results from the area of group dynamics provides guidelines for the leadership style of the teacher and for the facilitation of the interaction of the members of the small work groups.

Chapter IV is a description of the one-year pilot study which was conducted with a beginning college calculus course. The chapter includes an anecdotal record of certain significant classroom experiences pertaining to the dynamics of the learning group.

Chapter V describes the interaction between the students and the mathematics content during the pilot study.

Chapter VI is an evaluation of the pilot study, including data on student achievement, and a survey of student attitudes.

Chapter VII includes conclusions and a number of questions for further investigation.

1.3 OUTLINE OF CHAPTER I

The main body of Chapter I begins with a survey of current issues in calculus instruction. There follows a brief discussion of studies dealing with the presentation of the content of the first-year calculus course. A review is presented of five existing methods of calculus instruction, and a critical appraisal of the existing methods of instruction leads directly to the problem for investigation in this paper.

1.4 PRELUDE

Concern about the teaching of calculus is not a new phenomenon among mathematics educators. As early as 1894 Fletcher Durell stated recommendations for the presentation of topics in the differential and integral calculus. It seems likely that the discussion of the pedagogy of the calculus began before 1894, but it has certainly continued since that time.

During the sixth and seventh decades of the twentieth century, a strong interest in science and technology has prevailed in the United States. The scientific-technological emphasis has naturally led to a demand for better preparation of students in mathematics from the elementary grades through the college level. The subject of calculus in particular has been the focus of a great deal of attention.

There has been a substantial increase in the number of students taking courses in calculus; this increase has been attributed to several factors. The usefulness of calculus in the physical sciences and in engineering has long been clear; however, the need for calculus as a tool in the behavioral and the biological sciences has become apparent only in recent years. In addition, the use of high speed computers has been extended to deal with certain problems involving limit processes. Finally, the improved preparation of many mathematics students at the secondary level has had an effect. Numerous freshmen have been prepared to study calculus in order to fulfill a college mathematics requirement, and there has been a considerable increase in the number of students having an interest in mathematics as a career (Committee on Support of Research in the Mathematical Sciences 1968; hereafter cited as COSRIMS).

In addition to the burgeoning calculus enrollments in the colleges, there has been a trend toward the teaching of calculus at the secondary level. The advisability of teaching calculus to high school seniors has been the subject of considerable debate (Neelley 1961; Rising 1961; Brown 1961; Hildebrandt 1962; Allendoerfer 1963). Recommendations concerning the content of the calculus course and the training of high school teachers of calculus have been put forth by the Committee on the Undergraduate Program (CUPM) of the Mathematical Association of America (1966). The fact that many schools have not been able to carry out the CUPM recommendations has perhaps been reflected in the results of the Advanced Placement Examination in Calculus. Many students have taken the Examination, but there has not been general success in these attempts to achieve advanced placement (College Entrance Examination Board 1966).

The increased enrollments in the calculus have placed considerable strain upon the instructional capacities of the high schools, colleges, and universities. Shortages of space, money, and qualified instructors have awakened interest in several different methods of calculus instruction. The colleges and universities have in many instances turned to the large lecture system or to instructional television. There have been attempts to prepare programmed calculus textbooks (Kleppner and Ramsey 1965) and to develop computer-related instruction in calculus (CUPM August 1969).

The interest in various methods of instruction has been paralleled by innovations in the calculus content. There has been a trend toward the adoption of the integrated course in calculus and analytic geometry as a replacement for the two separate courses (Monroe 1966). There has also been a tendency away from the informal presentation of the calculus. Various modern textbooks have presented a highly formal, rigorous treatment of the subject, and many colleges have offered a freshman calculus course entitled "Analysis" (Rissler 1966).

In summary, the preceding discussion has pointed out the following major issues pertaining to calculus instruction: the difficulties of handling increased enrollments, the advisability and practicality of teaching calculus in the high school, the proper content and level of rigor of the calculus course, and the choice of an instructional method which maximizes learning.

The author has chosen to not directly address the problems of increased enrollments and calculus in the high school, to attend in a limited way to questions of content and rigor, and to concentrate upon the existence of an improved method of calculus instruction for use at the undergraduate level.

Although there has been extensive debate over the merits and demerits of various methods of calculus instruction, there has been little objective evidence gathered to support the conflicting claims. The following review of the literature encompasses the relatively small amount of research pertaining to the teaching of calculus at the undergraduate level.

1.5 STUDIES IN THE TEACHING OF CALCULUS

There have been several investigations dealing with the presentation of the calculus content. Monroe (1966) taught analytic geometry and calculus separately and also taught the two subjects in an integrated course, with no significant differences in student achievement on examinations. Levine (1968) did a ten-week study of the theory-to-application approach to analysis as compared with the experience-to-theory approach. The results, which did not include a full statistical analysis, seemed to favor the experience-to-theory approach.

Several of the content investigations were not empirical studies. Rissler (1966) traced the historical development of the mean-value theorems and their applications. His examination of current textbooks distinguished between " . . . an analysis approach, a traditional but strong approach, and a less rigorous approach." Holden (1967) presented geometrical ways to motivate and make plausible certain basic theorems of the calculus. Levi (1963) developed and successfully taught a course which placed emphasis upon the class of real functions which can be represented by power series.

Recommendations for the teaching of the calculus involve both content and general methods of instruction. Studies have been conducted with various instructional approaches, including the lecture, programmed instruction, non-programmed self-pacing methods, small group instruction, and discovery learning.

The Lecture

The lecture has been a widely used form of calculus instruction, and several studies have dealt with the lecture as presented to classes of various size or via television. In an experiment which dealt with the size of lecture classes, Stockton (1960) found that student achievement on examinations in a class of ninety did not differ significantly from achievement in a class of thirty.

K. May (1962) reported a two-year trial of the lecture-discussion system in calculus, in which students met for 3 periods per week in large lecture classes with 250 members and for 2 periods per week in small discussion sections with 15-20 members. Although student achievement was not decreased under the lecture-discussion system, both the students and the staff felt that small classes were more enjoyable. After the two-year trial period, the staff decided to discontinue the use of the large lectures. "The majority preferred to have a slightly heavier load accompanied by the satisfaction of managing a small class completely" (K. May 1962, p. 433).

Dyer-Bennett et al. (1958) found no difference in student achievement under large-lecture methods and closed-circuit television presentations. However, a survey of student attitudes showed a strong negative reaction to the television calculus. Students believed that the television lectures were less interesting than the live performance.

In the television situation, according to the results of the survey, there was no opportunity to get to know the instructor and no chance to ask questions. Moreover, the students generally concurred with a statement that television does not belong in the classroom.

Programmed Instruction

One alternative to the lecture method in calculus is programmed instruction, which permits each student to control the rate at which he learns the material. Programmed instruction in the calculus has been the subject of three studies. T. Davis (1966) compared a programmed approach with the lecture approach in the presentation of certain analytic geometry topics which often appear at the beginning of an integrated course in analytic geometry and calculus. Stannard (1966) compared the achievement of lecture students who used supplementary programmed materials with the achievement of students who did not have access to such materials. Shelton (1965) presented the limit concept through programmed materials with an abstract, deductive approach and also with a concrete, inductive approach. No significant differences in student achievement on examinations were found in any of the three studies.

Non-Programmed Self-Pacing Instructional Methods

Student self-pacing in the learning of calculus can take place without programmed instruction. Baum (1958), in an investigation of the feasibility of independent study, used the "usual classroom methods" during the first two-thirds of the academic year. During the final third of the year, members of the experimental group studied two texts on their own, while the rest of the students continued with classroom instruction. There was no significant difference in achievement of the independent study group and the classroom group.

Kingsbury (1963) let each student learn calculus at his own rate by working through a sequence of problem sheets, some of which were done in class and some at home. According to his own preference, the student could at any time read a textbook, consult with other class members or with the teacher, or work the problems by himself. For each problem sheet, there was a student "corrector" who would examine all the solutions in the light of an answer sheet prepared by Kingsbury. Correct answers were noted on a master chart, but incorrect papers were returned to the student, who could, if he wished, discuss his mistakes with the corrector. The student was expected to redo his erroneous solution, resubmit it to the corrector, and repeat the process as many times as necessary until the solution was finally right.

Kingsbury went to great lengths to establish personal communication between himself and his students. He solicited, and responded to, both verbal and written comments dealing with student complaints, suggestions for improvement, fears, and personal difficulties. He concluded, "In the end I can say that I have never been more warmly received by any class" (Kingsbury 1963, p. 10). The achievement of his experimental group on the final examination was significantly better than that of the control sections, which were instructed by the lecture method.

Discovery Learning

Two studies have dealt with discovery learning in calculus; the first of these was conducted by Cummins (1960). A "student experience-discovery approach" was used to teach a one-quarter beginning calculus course. " . . . There was constructed a series of study-guide sheets containing questions, suggestions, and exercises by which the student, either independently or with the help of class discussion, could arrive largely by his own efforts at some methods and facts of the calculus" (Cummins 1960, p. 163). Students used a textbook as a source of problems and a reference with which to compare their own results. Each student kept his own notebook, which was organized as he preferred.

The control sections were taught by traditional methods, and no significant differences in student achievement occurred with standard problems and manipulative skills. However, the experimental group exhibited greater understanding of the theory and logical structure of the calculus.

In describing the response of the class to discovery learning in calculus, Cummins has presented a number of remarkable quotations written by students. One of these states: "This approach makes the mathematics 'alive.' It is living and growing! Some students I talk to just differentiate--they do not know what they are doing, nor do they understand what a derivative is" (Cummins 1960, p. 169).

The second study related to discovery learning in calculus was done by Larsen (1961). The study included a discussion of heuristic ideas stemming from the works of Poincaré, Polya, and other authors. In addition, certain heuristic suggestions were written for student use outside of class.

Two experiments were conducted by Larsen, the first of which did not conclusively show that heuristic ideas could be learned in addition to the usual content of the course. However, " . . . it did produce some evidence that students can learn to use heuristic ideas when taught from the heuristic standpoint, but indications were that such learning may come at the expense of normal course content" (Larsen 1961, p. 2632). The second experiment showed that the presentation in written form of a particular, relevant heuristic suggestion could significantly reduce the time needed by students to solve the problem at hand.

Small Group Instruction

Small group methods provide an additional approach to calculus instruction. Limited use of small groups was made by Turner et al. (1966), who taught calculus for three days per week in large classes via the lecture-discussion method.

> For the remaining two days, the class was divided into small groups of twelve to twenty students with a senior mathematics major acting as a teaching assistant and in charge of each group. These small groups were further divided into groups with three students working together. The mathematics major was instructed to use a variety of teaching methods. The regular instructor was free to move from group to group working with individuals as he felt necessary. [Turner et al. 1966, p. 769]

The control section had the lecture-discussion method for five days per week. There were no significant differences in student achievement on examinations between the experimental section and the control section.

Summary of Noteworthy Results

In the studies dealing with various methods of calculus instruction, differences in achievement or time-to-criterion have attained statistical significance in three instances, namely, the self-pacing method of Kingsbury (1963) and the experiments of Cummins (1960) and Larsen (1961) in discovery learning and problem solving.

With the exception of the Dyer-Bennett study (1958) which showed strongly negative student attitudes toward television lectures in calculus, the experiments have not included statistical data on attitudes. Nevertheless, K. May (1962) has reported the abandonment of large lectures after a two-year trial period on the sole grounds that small classes were more enjoyable for everyone concerned. Both Kingsbury (1963) and Cummins (1960) have commented upon the strongly favorable responses of many students involved in the respective experimental programs.

1.6 A BASIS FOR THE DISCUSSION OF INSTRUCTIONAL METHODS

When considering the effectiveness of various methods of instruction, it is not possible to rest one's case exclusively upon measures of student achievement on examinations. Measures of student attitude, for example, provide another dimension for the assessment; these measures may refer to student attitudes toward either the mathematics content or the method of instruction. A third dimension for the assessment is obtained by measuring the time required to reach a specified criterion (Larsen 1961).

It is possible to view any system of instruction as a "black box." The students enter the box, undergo a certain instructional treatment, and emerge at some later time--possibly changed in some way. From this perspective, post-treatment measures of achievement, attitude, or time-to-criterion are measures of output which partially describe the results of the treatment applied in the instructional box. A decision about the effectiveness of an instructional treatment should not be made solely on the basis of a partial description of the results of the treatment. Accordingly, it is essential not only to gather data about the output of the instructional box, but to open up the box, look inside it as it operates, and carefully examine the nature of the instructional process which takes place therein. This examination-of-process approach is taken throughout the discussion which follows of various methods of calculus instruction.

1.7 DISCUSSION OF VARIOUS METHODS OF INSTRUCTION IN CALCULUS

The Lecture

The lecture in calculus is a discourse delivered by a lecturer in order to convey certain information to an audience of students. The discourse is usually prepared by the lecturer in advance of his presentation. The lecture in calculus also includes a written presentation by the lecturer--usually through the use of a blackboard or an overhead projector. The students in the audience listen to and watch the presentation; they frequently take notes in order to retain a record of the information presented by the lecturer.

The lecture method of instruction--when it works properly--has certain strengths. The method can be efficient; it is possible for the lecturer to cover a large amount of information in a relatively short amount of time.

In addition, the lecturer who is so inclined can present information gleaned from several different sources and can offer a unifying viewpoint which would be difficult for students to obtain elsewhere. The mathematics lecturer can comment upon the value and importance of the material which he presents, and he can, from the standpoint of his professional expertise, illustrate how an experienced mathematician and teacher thinks about the subject matter. The skillful lecturer can often make a clear, well organized, and interesting presentation of the material in the calculus course.

The preceding discussion describes the lecture at its best; in practice, however, in large classes or in small ones, the lecture often fails to promote student interest in the subject. Penalties may be instituted to reduce the rate of non-attendance, but boredom and lack of attention are frequently found among those who are physically present (Arnstine 1967).

Learning is said to be an active process, but the lecture often calls for no more than passive absorption of information. There is little in the lecture situation to guarantee any mental activity on the part of the student, who is often reduced to the role of a noncomprehending copyist. Reluctance to ask questions is prevalent, for the learner does not wish to appear ignorant or stupid in interrupting the steady flow of information (Saaty 1966).

The one-way communications pattern of the lecture provides almost no opportunity for feedback from the audience. The lecturer may be totally out of touch with the levels of student interest and comprehension of the subject matter, and the student who once "gets lost" may stay that way for the remainder of the period. In addition, the one-way communications pattern provides no opportunity for students to learn from one another, and no opportunity for anyone to hear ideas other than those of the lecturer. Finally, in the lecture situation neither the students nor the lecturer become known to one another as individuals.

The lecture method of instruction has both strengths and severe limitations. These limitations are to some extent reduced by certain variations of the lecture method, which are frequently found in calculus instruction.

Variations on a Lecture: Part I

A number of colleges and universities use the lecture-discussion system of calculus instruction. In this system there is a mass lecture on two or three days per week and smaller sections, led by the professor or frequently by teaching assistants, which meet for two additional periods per week. The meetings in the smaller sections are intended to serve the following purposes: to provide an opportunity for student participation in class, to clear up misconceptions about the materials covered in the lecture, to present solutions to homework problems, and to provide some opportunity for personal contact with a teacher.

The four goals of the discussion section appear to be quite important. Few people would argue, however, with the assertion that these important purposes could be better served in a small class which met daily with a good teacher. Indeed, the lecture-discussion system has been defended largely on the grounds that the university cannot afford to hire enough good teachers to handle the increased enrollments in calculus with small classes. Since the subject of this paper is the development of an improved method of calculus instruction, it seems appropriate to consider approaches other than the lecture-discussion system.

Variations on a Lecture: Part II

In some high schools, colleges, and universities, there are calculus classes of reasonable size--ranging, for example, from twenty to thirty students--which are taught in a self-contained mode. The teacher of such a class frequently uses one of several variations of the lecture method. One teacher might permit questions during the lecture only at specified times, such as at the end of a proof or problem solution. Another teacher might allow his students to ask questions at any time during the lecture. In some classes the student must raise his hand and be called upon before speaking, whereas in other classes a student may speak up when he wishes without prior permission.

In addition to permitting students to ask questions, certain teachers raise questions to be answered by the class members; this is done to encourage greater participation by students. The questions can be directed to the class as a whole or to specific individuals; again, there are possibilities for instructional variation. The teacher might ask a question every so often "just to keep the students awake and on their toes." The teacher might ask recall questions, which involve only the recollection of a specific item of information. In addition, the teacher might ask "higher order" questions, which entail the application of a principle or the formulation of a generalization (Teacher Education Task Force 1969). Finally, the teacher might direct a discussion in which the students provide the suggestions and reasons for each step in the problem solution, and in which the students are sometimes allowed to discuss the mathematics with each other. This last approach is more accurately described by the name "teacher-directed discussion" than by the phrase "variation of the lecture."

To what extent do the previously stated criticisms of the lecture apply when it is modified to permit student participation? The previously cited weaknesses of the lecture are largely connected with the one-way communications pattern. When student participation is permitted, the one-way communications pattern is no longer there, and the criticisms are either neutralized or else apply with lesser force. Stated in positive terms, it appears that students are more likely to understand the lecture when they have the opportunity to ask or to answer questions. Accordingly, the lecture which is modified to permit student participation may be rated as a better method of instruction than the lecture itself.

The operation of any variation of the lecture method is heavily dependent in practice upon the classroom behavior of the teacher. Accordingly, one significant route for the improvement of calculus instruction is the further development and implementation of programs for the training of classroom teachers at both the high school and undergraduate levels. The author--with full recognition of the importance of teacher training and with good wishes for the success of that approach--has chosen another route for the improvement of calculus instruction. The alternate route is the development of methods of instruction which greatly increase the participation of the learner, which depend less heavily upon a presentation by the teacher, and which place the teacher in a role basically different from that of a lecturer.

Programmed Instruction

Programmed materials are designed to maximize the participation of the student in the learning process, and such materials can be used as the core of a method of classroom instruction. Under this method, the student works through the programmed materials during the class period, and the teacher becomes the supervisor of the student efforts. In programmed instruction, the student not only participates actively in the learning process; he also controls the rate at which he learns the material. The student, then, is neither bored with a pace which is overly slow nor swamped by a pace which is overly rapid.

Unfortunately, the strengths of programmed instruction are countered by a number of weaknesses, which have been pointed out by Buck (1962) and by K. May (1964). Student boredom with programmed materials is a common phenomenon, which occurs despite the advantage of self-pacing. In programmed instruction, the student is presumed to be motivated by a drive to gain right answers, much as the hungry pigeon pecks for pellets. However, the knowledge that answers are correct is often not much motivation for human beings, particularly when the program is designed to produce a monotonous flow of ninety-five per cent correct responses. The very low expected error rate signifies that the student is able to work through the program with little thought. Intellectual challenge is absent; it is only necessary to follow the directions.

The objectives of programmed instruction are narrow. The objectives are limited by the format of the instructional program, which demands specific, short responses to limited, unambiguous questions. Insight into the solution of non-routine problems is replaced by mastery of skills and techniques. The skills are attained in the solution of routine exercises, and the value of the skills is not questioned in the program. Thus, the student learns how to apply a technique without asking why he does so or why it works.

In programmed instruction the advantage of self-pacing is attained at a high cost--the elimination of learning through communication with other persons. The student working through a programmed text by himself does not have to test his understanding through interaction with others. The student answers written questions but does not ask his own questions; he solves problems but does not set problems. He makes check marks or writes brief replies, without having to clearly summarize or present his ideas in either an oral or written form.

In summary, programmed materials offer the advantages of student self-pacing and active participation in the learning process. However, in programmed instruction, active participation occurs in the solution of routine exercises which demand little thought, and self-pacing is achieved at the expense of communication with other human beings. It appears that the programmed text may play an effective supplementary role, particularly in drill and practice (Buck 1962; K. May 1964), but it is necessary to look further for a basic method of calculus instruction. Perhaps there is a method of instruction which retains the advantages of student self-pacing and active participation, but which also promotes both thinking and interpersonal communication.

Structural Communication

A method of instruction named "Structural Communication" has recently been developed in England (Education Research Group 1967; hereafter cited as ERG). The method sometimes uses teaching machines "collectively named Systemaster." A complete communication cycle with the student involves the following four phases. (1) A message is presented to the recipient via television, a tape-recording, or other combinations of audio-visual media. (2) Certain problems or questions are posed, and the recipient is expected to provide an answer. The answer is selected from a "response indicator" which contains twenty numbered possible responses. (3) The pattern of numbered responses selected by the student is taken to be indicative of his grasp of the message. Depending upon the responses selected, certain other stimuli are presented to the student to reinforce particular aspects of the message. This is done in the form of a "dialogue," either with the machine or with a special textbook. (4) "After mistakes have been corrected, ideas amplified and understanding tested, it remains to bring the communication into one integral whole" (ERG 1967, p. 197). This is done by means of a final statement giving the "Viewpoints" of the author.

In certain respects the structural communication approach is similar to programmed instruction. The student uses either a teaching machine or a special textbook with "branching." In addition, the responses of the student are limited, definite, and unambiguous in character. However, in other respects the goals of structural communication are unlike those of programmed instruction. "In structural communication the aim is to evoke understanding, not to convey facts except as a by-product" (ERG 1967, p. 191). Moreover, the proponents talk a good deal about "dialogues" between the author and the student, the "total and integral meaning" of a communication, and of approaching "the problem of understanding in terms of creativity and consciousness rather than of conditioning and automatism" (ERG 1967, p. 186).

There has not been a report of the use of structural communication in calculus instruction, although the method has been used in teaching abstract algebra. The author is somewhat intrigued with the approach, although he has not seen it operate in practice. A cautious evaluation can be attempted, "sight unseen," on the basis of the four previously stated criteria of student active participation, self-pacing, thinking, and interpersonal communication.

It appears that structural communication shares the strengths of programmed instruction with respect to active participation and self-pacing by the student. The author hazards the guess that structural communication actually does promote more thinking by the student than does programmed instruction, despite the limited nature of the possible student responses. On the fourth criterion of interpersonal communication, structural communication does indeed involve a certain type of "vicarious" communication between the author of the message and the recipient. However, the element of face-to-face communication between human beings is missing, as in the case of programmed instruction. The basis for the author's emphasis upon interpersonal communication should become quite clear in the following section of the paper.

Loneliness, Anxiety, Alienation, and Mass Education

"The Loneliness and Anxiety of Modern Man" is a chapter title in one of the works of the psychotherapist

Rollo May (1953). According to this therapist, a basic "characteristic of modern people is loneliness. They describe this feeling as one of being 'on the outside,' isolated, or if they are sophisticated, they say that they feel alienated" (R. May 1953, p. 23). "Anxiety, the other characteristic of modern man, is even more basic than emptiness and loneliness. For being 'hollow' and lonely would not bother us except that it makes us prey to that peculiar psychological pain and turmoil called anxiety" (R. May 1953, p. 30).

The loneliness of modern man is a widespread condition, not a phenomenon restricted to a small proportion of the population. Sarnoff has described the infrequently realized hopes of individuals "for a moment of real attention from others, for a flash of genuine concern about themselves in the eyes of others. . . " (Sarnoff 1966, p. 186). Riesman (1952) has chosen a book title which speaks for itself--The Lonely Crowd.

The Association for Supervision and Curriculum Development (ASCD) has deemed it necessary to hold a conference to counter the forces in mass education which ignore "the person in the process" (ASCD 1967). Combs has written a somber introduction to the report of that conference. "We have loosed in the world great forces of alienation and depersonalization. We cannot afford to be made the victims of this imbalance. An educational system which dehumanizes its charges fails everyone, its students, its professional responsibilities and society itself" (ASCD 1967, p. vi).

Support for "the person in the process" has been voiced by certain members of the mathematical community. According to Saaty, "It is usually taken for granted that for producing worthy scholars, a good mind is most of what is needed and personal development is secondary. However, there are enough examples of brilliant individuals who failed because of lack of personal development that we need to re-examine this philosophy in order to decrease what I consider a substantial waste of talent, and a nontrivial loss" (Saaty 1966, pp. 398, 399). Saaty goes on to consider such matters as "the absence of needed personal warmth to temper an imagined or existing nonconcern or perhaps harshness in the environment," the pain involved in "being effective in a real world of people," the need for "courage and constructiveness in human relations," and the "distinction between assembly line production of talent and a respect for the higher and deeper value of young people" (Saaty 1966, pp. 398-400).

The forces toward loneliness, anxiety, and alienation are present in the modern university as well as in the society at large. Some of these forces and some of their effects are described in the following paragraphs.

The enormous increase in the enrollments of the universities has given rise to the regulation of students by an expanding bureaucracy. "The administration has become remote and impersonal, leaving students to feel themselves at the mercy of computers, and in danger of simply becoming a number" (Logan 1967, p. 336). "And it has certainly been made abundantly clear by all kinds of analysts that bureaucracy, the product of 'bigness,' both demands and contributes to standardization, replaceability and irrelevancy of human beings" (Lloyd-Jones 1967, p. 42).

The expansion of university enrollments has led to an increased use of mass lectures in various subject areas. According to the COSRIMS Report (1968), forty-five per cent of the universities having doctoral programs in the mathematical sciences used large lecture sections in elementary calculus to varying extents during the academic year 1965-1966. Large lecture sections were defined as sections having more than one-hundred members. Moreover, the COSRIMS Panel on Undergraduate Education has recommended an increase in the use of large lecture sections.

The increased use of mass lectures is not confined to the mathematical sciences but is a general trend in university instruction. "There is an increasing deterioration in the relationship between the freshman and the experienced professor, the master teacher, the established scholar, the practiced classroom educator. . . . The professor is relegated to the giant lecture section where he talks . . . to two or three hundred anonymous students" (Rapport and Goldman 1967, p. 123).

In the period 1962-1965 a formal teaching load of six hours per week was "typical" for three-hundred distinguished American mathematicians (COSRIMS 1968). Moreover, younger faculty members in many fields have become increasingly interested in funds for research, reduced teaching loads, grants, and leaves of absence. Many of these new professors have begun to regard themselves "primarily as scholars" (Hacker 1967).

Numerous students at the undergraduate level have not felt pleased by their dual exposure to research-oriented scholars and a "remote and impersonal" administration.

"This state of affairs explains the widespread student demands for more voice in the appointments and promotion of faculty, because they see good teachers being let go and indifferent teachers retained under a 'publish or perish' criterion" (Logan 1967, p. 336).

The importance of grades with respect to the attainment of honors, awards, scholarships, fellowships, and admission to graduate schools has become "the central reality of academic life" for many undergraduates (Katz 1968, p. 24). In addition, the trend in numerous colleges and universities has been toward higher and higher standards for admission, retention, and honors. "The final object of these pressures, the student, is cast, whether he likes it or not, into what we have called the classroom drama of competition . . . . With every other student becoming a potential threat to each student for the available rewards and space, the Lonely Student is emerging" (Rapport and Goldman 1967, p. 125).

There has been a marked increase in the student need for psychiatric or psychological counseling services on the major campuses; this need has been partially met with substantial increases of professional staff for the various counseling centers. Although it is not at all clear that the university environment can in itself create emotional problems, it appears that the university acts as a "massive trigger" which sets off previously existing student tendencies toward worry and anxiety (Lono 1967).

"Two out of every ten American students need psychiatric help during those four bright college years once thought of as the best of our lives. About half of them get it. Colleges with mental health facilities find the psychiatrists are getting busier and busier" (Jackson 1967, p. 140). Students come to the psychiatric clinics with a number of complaints, including depression, inability to do schoolwork, nervousness, tenseness, confusion, psychosomatic problems, and strong anxiety (Ellis 1968). Although in many instances the students seek and receive help through counseling, the problems of certain students are ended only by suicide (Lono 1967).

While many students blame themselves for their problems, other students place the blame upon the college or university environment. "In Peterson's . . . 1964-1965 survey of 849 accredited four-year institutions, 27 per cent of the colleges reported that student protest had taken place over issues involving courses, tests, and grades"

(Axelrod 1968, p. 119). These protests were separate from civil rights protests, which were current at that time.

In conclusion, the university environment can hardly be described as warm, supportive, or therapeutic in character. Therefore, the designer of a method of mathematics instruction for use at the college or university level would be well advised to take into account some of the affective needs and problems of students. Although it would be naive to expect a method of mathematics instruction to resolve the emotional problems of students, it might be possible to design a method of instruction which does not aggravate or accentuate those problems and which, by its very nature, provides support for "the person in the process."

Small Group Instruction

The use of small group instruction may be regarded as an attempt to counter the forces toward anonymity, loneliness, anxiety, and alienation which are present in the modern university. In a group with at most five members, the student engages in regular face-to-face discussions with a small number of his colleagues. The number of students per group is deliberately restricted so as to increase the possibilities for personal contact and for active participation in the discussion.

The case for small group instruction rests partly upon the importance of the affiliative needs of human beings. The existence of a social impetus toward human activity has been emphasized by Dewey (1916, 1938) on philosophical grounds. On scientific grounds the importance of social interaction has been stressed by Montagu.

> The identity of the person consists of and is derived from the meaningfulness of his interrelationships.
>
> In short, the physiological dependency of the fetus and the newborn becomes, in society, a socially organized dependency, a social dependency in which the interacting person finds the meaning of his life in his relations with other persons and their thoughts and activities. Unheeded, the physiologically dependent infant would die. Unheeded, the socially dependent adult falls into an apathy which may lead to death. [Montagu 1966, p. 75]

In support of his conclusions, Montagu has cited the works of Adler, Erikson, Fromm, Freud, Sullivan and numerous other psychiatrists, psychologists, anthropologists, biologists, and sociologists.

The idea of small group instruction may arouse immediate doubts about conformity pressure and submergence of the individual's identity in the group. There are, moreover, the possibilities that the small group may increase anxiety through intense competition and may increase loneliness through rejection of an individual. The fears of conformity pressure, anxiety, and rejection in small groups are not without basis in reality, since small group meetings may indeed have any or all of these effects. On the other hand, a good deal is known about the facilitation of small group interaction (Cartwright and Zander 1960). This knowledge of group dynamics can be applied by the teacher who supervises the small groups, in order to create a friendly and supportive group atmosphere in which conformity pressure and anxiety are minimized and in which independent thinking and mathematical productivity are maximized (see Chapter III).

The case for small group instruction can also be made on mathematical grounds. The importance of student discussion in mathematics education has been emphasized by Buck.

> . . . Let me remind you that student-student interactions are also important in learning, and that at the professional level, much mathematical research springs from discussions between mathematicians. Moreover, a test of understanding is often the ability to communicate it to others; and this act itself is often the final and most crucial step in the learning process. [Buck 1962, p. 563]

It appears to the author that student-student interactions can most readily take place within a small-group setting.

In the Nuffield Project, elementary school children have learned mathematics in small groups, with three members apiece (R. Davis 1967). At the undergraduate level, Turner et al. (1966) have used small group instruction in calculus as a supplement to several large lectures per week. In view of the previous discussion, the author suggests the use of small groups as a replacement, rather than a supplement, for the lecture in calculus.

When the small group method is applied in calculus instruction, there is room for variation in the approach to the mathematics content. At least two approaches are possible, and these may be described respectively as activity learning and discovery learning. In support of the distinction, one notes that student discovery is one among many possible responses in an activity, but that the activity may take place without any discovery, as in the case of programmed learning. Hence, discovery learning is one form of activity learning--a form in which the students discover various mathematical facts or relationships. There are other forms of activity learning in which the students solve routine exercises without making any discoveries. There now follows a discussion of these alternate approaches to learning the mathematics content.

A Non-Programmed Self-Pacing Method of Instruction

In the self-pacing activity method of Kingsbury (1963), students learn by working through problem sheets at their own rate in consultation, when desired, with other class members, the teacher, or a textbook (see p. 11). The textbook presents the theory and the techniques for solving various classes of problems, and the students gain mastery of the material by working through the problem sheets related to the readings, by handing in their work for correction, and by redoing all erroneous solutions.

In the opinion of the author, the Kingsbury method requires an intermediate level of student thought. There is substantially more thought required than in the step-by-step prescriptions of programmed instruction, but there is less intellectual challenge than in many of the nonroutine problems of a discovery learning approach to calculus.

In summary, the Kingsbury method offers a combination of student self-pacing and active learning with some social interaction and an intermediate level of thinking. When examined from a perspective which emphasizes all four of these variables, the Kingsbury method stands out as a substantial step in the improvement of calculus instruction.

The author has only one basic suggestion for improvement of the Kingsbury method. Under this method the goal of the student is to complete as many of the problem sheets as he can; his progress toward the goal is reduced if he spends time helping someone else. Thus, there are the dangers of hesitancy in asking for needed assistance and resentment in giving assistance. Moreover, the instructor cannot offer much assistance to any one student unless the size of the class is quite small.

It appears that the tension involved in giving and receiving help could be relieved if the students began to work together in small groups to provide mutual assistance and intellectual stimulation. The small-group structure could be instituted right at the start of the class, in order to avoid the social discomfort and inefficiency involved in spontaneous group formation over a period of time. The goal of the groups could be to solve the problems in such a way that the solutions were understood by all the members, in which case it would not be necessary for every student to hand in every problem for correction. It would seem that the advantages of self-pacing could be retained to a large extent within the small-group structure, since each group would set its own pace in accordance with the needs of its limited membership.

Learning by Discovery

In his famous "principle of active learning," Polya asserts: "For efficient learning, the learner should discover by himself as large a fraction of the material to be learned as feasible under the given circumstances" (Polya 1966, p. 125). Polya (1966) has advocated both active participation and thinking by students, often in the search for solutions to nonroutine problems.

Moise has commented:

> . . . I believe that [R. L.] Moore's work proves something of broad significance. It proves that sheer knowledge does not play the crucial role in mathematical development that most people suppose. The amount of knowledge that a small class can acquire, struggling at every stage to produce its own proofs, is quite small. The resulting ignorance ought to be a hopeless handicap, but in fact it isn't; and the only way that I can see to resolve this paradox is to conclude that mathematics is capable of being learned as an activity, and that knowledge which is acquired in this way has power which is out of all proportion to its quantity. [Moise 1965, p. 409]

Discovery Learning via the Moore Method

Moise has described the operation of the Moore method. "He [Moore] would present postulates, definitions, and propositions; and the student's job was to find out which of the propositions were true, and present proofs or counterexamples" (Moise 1965, p. 408). Each student was to carry on his own investigation outside of class under the injunction that both reading and discussion of the subject with anyone else were forbidden. For each problem, a presentation was made during the class period by one of the students who claimed to have achieved a solution. The proposed solution was examined by the class members and by the professor with the intention of exposing any errors or gaps in the reasoning.

A search of the literature has revealed no reference to the teaching of calculus by the Moore method; therefore, one can only speculate about the possible results. If the method were applied in a calculus class of very bright students, one might anticipate the following outcome: a small number of the students would experience an enormous growth in their mathematical prowess and might go on to become productive, or even famous, mathematicians, as have some of Moore's astonishing topology students.

A further comment is in order, in view of the fact that the Moore method engenders intense competition and enormous achievement pressure, and also forbids communication among the students. These factors amount to a potentially devastating prescription for heightened anxiety and for increased isolation from one's fellows. Accordingly, the author does not recommend the use of the undiluted Moore method for calculus instruction.

For purposes of calculus instruction, it may be possible to make use of the procedure of Moore in which the instructor presents definitions and propositions and in which the students provide proofs or counterexamples. It appears that the challenging problems and the intellectual excitement could be retained, while the social patterns were changed in order to alleviate anxiety and to promote interpersonal communication.

A "Definition" of Discovery Learning

There is no single, generally accepted definition of discovery learning. For purposes of calculus instruction, the author has chosen to define a discovery learning approach by the following attributes:

1. General problems arise from "natural situations" (Cummins 1960).

2. Concrete experience precedes abstraction (Cummins 1960).
3. The students themselves do the mathematics.
4. The teacher provides limited guidance, often in the form of heuristic suggestions (Larsen 1961; Polya 1965).
5. The problems should be meaningful, relevant, and interesting to the students (Polya 1965).
6. Many of the problems are of a challenging, non-routine nature (Polya 1965; Moise 1965).
7. Some of the problems entail the exploration of open-ended situations (Pollak 1966).
8. Practice with manipulative skills occurs "naturally, but incidentally, in solving problems which are primarily designed to broaden understanding and to stimulate creative thinking" (Dilworth 1966, p. 92).
9. " . . . The students actively contribute to the formulation of the problem that they have to solve afterwards" (Polya 1965, p. 105).
10. For each proposition under consideration, the students provide either a proof or a counter-example (Moise 1965).

11. The students develop the techniques for solving various classes of problems.
12. There is no textbook in the usual sense, but there may be a book with definitions and problems.

Three Discovery Learning Approaches

Positive results in calculus instruction have been attained via the "student experience-discovery approach" of Cummins (1960) and via the heuristic problem solving approach of Larsen (1961). Cummins, in particular, has made use of class discussion to facilitate the discovery process. Hence, discovery learning may take place through a teacher-directed discussion of the entire class or through individual work, as in the Moore method. There is a third alternative, the "in-between" case, in which discovery learning takes place with small groups of students who work together to solve the problems at hand, with limited guidance by the teacher.

Discovery in small groups, as compared with teacher-directed discovery by the entire class, would permit more active participation by the students; this is because the opportunity for participation increases as the group size decreases. Discovery in small groups, as compared with individual discovery, would make use of the social impetus for human activity and would provide both increased intellectual resources and emotional support from the peer group. Hence, it appears that the small group approach would make it possible for the students to tackle some challenging problems, but without the previously cited dangers of isolation and anxiety.

Small Group Activity or Small Group Discovery?

The small group method of calculus instruction might be used in conjunction with either activity learning or discovery learning. In a small group-activity approach, there would be a textbook to present the theory and the techniques for solving various classes of problems. The students would read and discuss the material in the book and would gain mastery by working problems related to the readings. In a small group-discovery approach, there would be no textbook in the usual sense. The students themselves would prove the theorems and develop the techniques for solving various classes of problems. In addition, there would be considerable exploration of open-ended situations.

It appears to the author that the small group-activity approach might be suitable for use with a large general population of calculus students. On the other hand, the rigors of the small group-discovery approach might restrict its use to a small population of bright students who were seriously interested in mathematics. To meet the needs of the different populations, it would be necessary to develop both the activity approach and the discovery approach; however, this could not be done in a single pilot study.

In selecting only one approach for development, it helps to temporarily disregard the question of instruction and to think instead about curriculum development. An experiment by Kaplan (1964) has shown the existence of a discrepancy between the order in which students prefer to receive information and the order in which the information is presented by teachers and textbooks. In the experiment, students generated learning sequences which were considerably different than the sequences presented by several textbooks to cover the same material about vectors. According to Kaplan, "Instructor-generated sequencing, which controls the learner, demands that a student learn according to a strategy that may not be his. If the strategy is not his, then it may not be meaningful to him" (Kaplan 1964, p. 298).

In view of the possible discrepancy between student-generated and instructor-generated learning sequences, it would be premature to develop the small group-activity approach in calculus; the author might unintentionally design learning sequences which were not appropriate for students. On the other hand, the exploratory problems of the discovery approach, in conjunction with the opportunity for student formulation of problems, would reduce the imposition of "artificial" learning sequences upon the students; these practices would instead let the students attack the material in their own ways. Through observation of the students' efforts in the small discovery groups, much might be learned about the ways in which students think about the subject matter of calculus.

It appears, then, that the small group-discovery approach could be used for both instruction and curriculum development. By using the information obtained through the discovery approach, it might be possible to specify the learning sequence for the activity approach in accordance with the preferred learning styles of students.

1.8 STATEMENT OF THE PROBLEM

The criticisms of the lecture and of programmed instruction have led to the following question: Does there exist a method of calculus instruction which simultaneously promotes student self-pacing, active participation, thinking, and interpersonal communication?

The criterion of interpersonal communication has been emphasized in order to promote student discussion of mathematics and to counter the forces toward loneliness, anxiety, and alienation in the modern society in general and in the university in particular. The small group method of instruction has been chosen to facilitate interpersonal communication, and it appears that the small group method also fosters active participation and, to a large extent, self-pacing by students. Moreover, small group instruction might be used in conjunction with either activity learning or discovery learning, depending upon the desired level of student thinking. Finally, it appears that small group instruction in conjunction with discovery learning might be suitable not only for instruction but also for curriculum development, through the observation and identification of student learning styles in calculus.

The literature contains two reports of discovery learning experiments in calculus, both with positive results, and one report of a limited use of the small group method. The combination of these two approaches has apparently not been attempted.

The problem for investigation reported in this paper has two parts:

1. to design a new strategy of mathematics instruction in which students work together in small groups to discover mathematical results
2. to put the strategy into operation for a period of one academic year with a beginning college calculus class.

The new strategy is named the "small group-discovery method" of mathematics instruction. The design of the instructional strategy is the subject of Chapters II and III.

CHAPTER II

JOHN DEWEY AND THE SMALL GROUP-DISCOVERY METHOD

2.1 OUTLINE OF CHAPTER II

This chapter includes a partial description of a systematic philosophy of education--John Dewey's educational philosophy of Experimentalism. The first part of the chapter begins with a brief discussion of the importance of the Deweyan philosophy and of its possible utility in mathematics education. There follows a description of those aspects of the philosophy which can be directly used for the design of the small group-discovery method. The actual design of the method on the basis of the Deweyan philosophy is the subject of the second part of the chapter.

PART I: A PARTIAL DESCRIPTION OF THE DEWEYAN PHILOSOPHY

2.2 THE INFLUENCE AND UTILITY OF DEWEY'S PHILOSOPHY

George Polya has emphasized the importance of student thinking, active learning, discovery learning, and interest in mathematics.

1. Thinking. "It is my personal conviction that the principal task of mathematics instruction on the high school level is to teach those young people to THINK" (Polya 1966, p. 124).

2. Active Learning. "Mathematics is not a spectator sport: it cannot be enjoyed and it cannot be learned without active participation. . . " (Polya 1966, p. 125).

3. Discovery Learning. "For efficient learning, the learner should discover by himself as large a fraction of the material to be learned as feasible under the given circumstances" (Polya 1965, p. 103).

4. Interest. "The best motivation is the student's interest in his task" (Polya 1965, p. 105).

It has not been widely recognized in the mathematical community that the emphasis on student thinking, active learning, discovery learning, and interest are particular aspects of a general philosophy of education and of life. It appears that many ideas for use in mathematics education might be found by turning to the general philosophy itself. Moreover, the emphasis in this philosophy upon learning as a social process has stimulated a substantial amount of research in the area of group dynamics, which is reported in Chapter III. The philosophy in question has become known under a confusing variety of names, including "Experimentalism," "Instrumentalism," and "Pragmatism"; its foremost advocate in education was John Dewey.

The views of Dewey and of his "followers" in the progressive education movement have had a profound impact upon educational theory and practice in the United States--an impact which has not been of a completely beneficial nature. Various scholars (Cremin 1966, Archambault 1966) have stated that the views of Dewey were often misunderstood, distorted, or over-simplified, not only by his critics, but by many of those who claimed to be his followers. Moreover, according to Archambault (1966), appraisals of the work of Dewey have tended to divide into two categories, namely "laudatory appreciations" and "polemical attacks." There follows an example from each of the two categories.

"As to Dewey's comparative place in the history of philosophy, I place him next to Plato and Aristotle. As to his place in the history of philosophy of education, he is, as I see it, the greatest the world has yet beheld. . . " (W. Kilpatrick 1966, p. 6).

"Educators, parents and students must be continuously stirred up by the defects in our educational system. They must be induced to abandon the educational path that, rather blindly, they have been following as a result of John Dewey's teachings" (Eisenhower 1959).

For this paper, there has been no attempt to undertake a comprehensive survey of Dewey's published works, which total several thousand pages in length. Instead, the author has chosen to base the present chapter on two of Dewey's most important volumes--Democracy and Education (1916) and Experience and Education (1938). Cremin, who has written extensively on the history of the progressive-education movement, has described the two volumes as follows: Democracy and Education " . . . wove the diverse strands of a quarter-century of educational protest and innovation into an integral theory" (Cremin 1966, p. 14). The second volume " . . . was destined to be his most important pedagogical work of the thirties, Experience and Education. The volume is really a restatement of aspects of his educational outlook in the context of the criticisms, distortions, and misunderstandings which had grown up over two decades" (Cremin 1966, p. 21).

The following discussion is a partial description of Dewey's philosophy, based upon two of his most important works. In the description of the philosophy, attention is limited to those aspects which can be directly applied in the design of the small group-discovery method.

2.3 EDUCATION AS AN ACTIVE, SOCIAL, PERSONAL EXPERIENCE

Education as Experience

Education is to be based upon personal experience of the learner (1916, 1938).* It is often assumed to the contrary that personal experience is not necessary and that knowledge can be communicated directly to the learner. However, the attempt to communicate knowledge directly may fail to take into account the meaning and the significance of the material dealt with.

> An ounce of experience is better than a ton of theory simply because it is only in experience that any theory has vital and verifiable significance. An experience, a very humble experience, is capable of generating and carrying any amount of theory (or intellectual content), but a theory apart from an experience cannot be definitely grasped even as theory. It tends to become a mere verbal formula, a set of catchwords used to render thinking, or genuine theorizing, unnecessary and impossible. [1916, p. 144]

Experience and education are not identical, for some experiences are "educative" and others are "mis-educative"

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*Throughout this chapter the references "(1916)" and "(1938)" are used as abbreviations of the references "(Dewey 1916)" and "(Dewey 1938)."

(1938, p. 25). Any experience which tends to hinder the development of further experience is said to be mis-educative. For example, an experience might produce mechanical skill at the expense of thought or, perhaps, promote an attitude of insensitivity or indifference toward learning. Hence, it is not enough that the student have an experience; what counts is the quality of the experience (1938).

An educative experience engages the student's interest in an active and voluntary fashion; however, it is not sufficient that the experience be enjoyable on its own terms. In addition, experiences should be connected with one another in a cumulative manner so that the experience of the present "lives on" with a beneficial effect upon the experience of the future (1938).

The Environment

Personal experience does not occur in total isolation but in a certain medium, or environment (1916, 1938). The individual deals with certain objects or ideas or other people, which constitute his environment and define his reality. If the person tries to do something, environmental conditions facilitate or hamper his attempts. When education is based upon personal experience, the individual learns by means of interaction with his environment. The teacher does not teach the student directly, but instead structures the environment in which the learning is to occur (1916, 1938). The environment is structured deliberately by using certain physical objects, sets of ideas, or social conditions, depending on the particular situation and the desired goals.

Activity versus Passivity

Interaction with the environment is an active process, rather than a matter of "passive absorption." The person actively does something, and in the course of his activity he learns. "It is not the business of the school to transport youth from an environment of activity into one of cramped study of the records of other men's learning; but to transport them from an environment of relatively chance activities (accidental in the relation they bear to insight and thought) into one of activities selected with reference to guidance of learning" (1916, p. 274). The learning in itself is not the objective for the individual; instead, learning occurs as a by-product of participation in some activity having a purpose.

The view of education as an active process is lent credence by the fact that healthy individuals under conditions of free choice do not tend to be passive. People of all ages have desires, impulses, and purposes, which lead them to participate in various activities (1938). There is, moreover, a natural tendency towards exploration and experimentation (1916). The tendency is particularly apparent in the case of infants, who constantly reach out to manipulate objects in their environment; however, the tendency remains to a greater or lesser extent at each stage of life. The job of the educator is to work in accordance with these natural impulses toward exploration. Education cannot then be regarded as a matter of "telling and being told" (1916).

Shared Activities

In addition to a physical environment, individuals have a social environment. However, the classroom situation frequently entails a socially limited environment in which acceptable behavior consists of silence or of limited communication with the teacher (1916, 1938). With the removal of this artificial restriction on communication, students have an opportunity to interact with each other as well as with the teacher. It is possible to form a social group as opposed to a collection of individuals (1916, 1938).

What would be the advantages in having a group, rather than a collection of individuals? In the first place, there is a social motive for much of human activity; individuals have a major interest in sharing their activities with others (1916). Furthermore, when a group decides to perform a task, the individual members may view the group success or failure as their own. The members are thus motivated to work toward the group goal, and the entire activity takes on a certain vital significance because of the common interest in the outcome (1916).

In the give-and-take of normal communication in the group, the material under discussion gains meaning (1916). The group members verbalize and clarify their own ideas in order to communicate with others, with the result that misconceptions can be quickly recognized and corrected. The experience of each member is enlarged by the contributions of the others, and the addition of new ideas serves to stimulate imagination. There are thus many benefits in a group situation where the learner becomes a "partner in a shared activity" (1916).

2.4 THINKING, NONTHINKING, PROBLEM SOLVING, AND SKILL FORMATION

Thinking versus Routine and Capricious Activities

An educative experience must involve thought. "The sole direct path to enduring improvement in the methods of instruction and learning consists in centering upon the conditions which exact, promote, and test thinking. Thinking *is* the method of intelligent learning, of learning that employs and rewards mind" (1916, p. 153).

Thinking is not likely to occur in an activity which is "routine" or "capricious" (1916, p. 77). The routine activity is automatic, repetitive, mechanical, or uniform. The routine activity increases skill along particular lines without leading to increased meaning or significance; the range of perception is narrowed, rather than widened.

In a capricious activity, the individual simply "lets himself go" without concern for what he is doing. He may be acting under the orders of someone else, and he follows the prescribed directions with no regard for the consequences or the significance of his actions. He acts thoughtlessly because the situation entails no purpose of his own (1916).

Thinking and Problem Solving

Thinking may occur in a situation if the individual has some purpose to accomplish and if the outcome of the activity is in doubt (1916, 1938). The doubt may exist because of some difficulty or obstacle inherent in the situation, in which case the individual has to find means of removing or circumventing the obstacle. The doubt may also occur because of the ambiguity or lack of clarity in the situation, in which case the lack of clarity may itself be regarded as an obstacle. In either case the individual has a problem at hand.

In problem solving which is intelligently directed, the person is trying to accomplish something. He has a goal or end in view, and he carefully selects means which will bring about the desired results (1916). He keeps in touch with the existing situation by gathering and evaluating data. He observes the consequences of his actions and modifies his behavior when necessary. He may find it necessary to formulate, refine, and test hypotheses. He is therefore engaged in a process of inquiry or investigation in order to resolve a doubtful situation (1916).

Thinking may or may not occur if the individual is presented with a problem to be solved; capricious activity, rather than thinking, may be the outcome. Hence, it is important that the individual be genuinely interested in solving the problem. "Is it the pupil's own problem, or is it the teacher's or textbook's problem, made a problem for the pupil only because he cannot get the required mark or be promoted or win the teacher's approval, unless he deals with it?" (1916, p. 155).

Care must be taken to see to it that the problem is of the right level of difficulty. The difficulty involved must be sufficiently great to arouse active thought, but should not be great enough to appear overwhelming (1916). The situation should be novel in certain respects, yet contain certain familiar points which connect with the individual's previous experience. Moreover, the situation would entail only routine activity if there were no possibility for error; a difficulty great enough so that mistakes may occur is a necessary condition for the involvement of intelligence (1916).

Thinking in the Formation of Habits and Skills

The formation of habits and skills is essential in order to deal with certain kinds of repeated situations.

However, under varying environmental conditions habits are helpful only when applied with thought. There is the danger that the habitual action remains the same while the environmental conditions change, thereby resulting in a lack of ability to cope with new situations. Accordingly, habits and skills should be formed under conditions where thought is necessary, rather than simply by means of routine and repetition (1916). Circumstances can be changed during the period of formation so that at each stage the student is not simply repeating what came before, but instead has to modify his action to meet the new situation. At the time the skill is being developed, the learner also becomes more disposed to use judgment in novel circumstances (1916).

2.5 KNOWLEDGE, DISCOVERY LEARNING, AND THE ORGANIZATION OF SUBJECT MATTER

Knowledge versus the Record of Knowledge

There exists a tremendous body of written subject matter. "The imposing stupendous bulk of this material has unconsciously influenced men's notions of the nature of knowledge itself. The statements, the propositions, in which knowledge, the issue of active concern with problems is deposited, are taken to be themselves knowledge. The record of knowledge, independent of its place as an outcome of inquiry and a resource in further inquiry, is taken to be knowledge" (1916, pp. 186, 187).

The distinction between knowledge and the record of knowledge is a crucial one. A person gathers, discovers, or creates knowledge in the course of some activity having a purpose; this active process is not the same as the absorption of the record of knowledge--the fruits of past activities (1916). When the record of knowledge is mistakenly taken to be knowledge, the acquisition of information becomes an end in itself, and the student spends his time absorbing what other people have done, rather than in having experiences of his own. The student is treated as a "piece of registering apparatus," which stores up information isolated from action and purpose (1916, p. 147).

The preceding discussion does not assert that informational knowledge has no value. Information has value indeed to the extent that it is needed in the course of some activity having a purpose, and to the extent that it furthers the course of the activity. "Informational knowledge is the material which can be fallen back upon as given, settled, established, assured in a doubtful situation. It is a kind of bridge for mind in its passage from doubt to discovery" (1916, p. 183).

Learning by Discovery

The distinction between knowledge and the record of knowledge provides the rationale for discovery learning. Rather than simply absorbing the record of other people's activities, the student is to be involved in the discovery or creation of knowledge in the course of purposeful activities of his own (1916).

> . . . (1) We are concerned with originality of attitude which is equivalent to the unforced response of one's own individuality, not with originality as measured by product. No one expects the young to make original discoveries of just the same facts and principles as are embodied in the sciences of nature and man. But it is not unreasonable to expect that learning may take place under such conditions that from the standpoint of the learner there is genuine discovery. While immature students will not make discoveries from the standpoint of advanced students, they make them from their own standpoint, whenever there is genuine learning. (2) In the normal process of becoming acquainted with subject matter already known to others, even young pupils react in unexpected ways. There is something fresh, something not capable of being fully anticipated by even the most experienced teacher, in the ways they go at the topic, and in the particular ways in which things strike them. Too often all this is brushed aside as irrelevant; pupils are deliberately held to rehearsing material in the exact form in which the older person conceives it. The result is that what is instinctively original in individuality, that which marks off one from another, goes unused and undirected. Teaching then ceases to be an educative process for the teacher. At most he learns

> simply to improve his existing technique; he does not get new points of view; he fails to experience any intellectual companionship. [1916, pp. 303, 304]

The Organization of Subject Matter

When the learner is engaged in a process of discovery, the subject matter of the expert, or specialist, in a particular discipline cannot be covered in a completely organized and systematic form (1916). It is necessary to take into account the tremendous gap in experience between the learner and the specialist. For the specialist, the logical organization of the discipline serves to clarify and to systematize material which is already familiar, meaningful, and significant. For the learner with limited experience, the same subject matter may appear to be remote, abstract, and unreal (1916). Accordingly, the kind of organization which clarifies matters for the specialist may function as an obstacle to the student. This is not to say that organization should be replaced by chaos; indeed, the goal is not to abolish the organization of subject matter, but to organize the material in a fashion most conducive to learning (1938). It must be emphasized that the environment can be educative only when matched to the purposes and capabilities of the learner.

When becoming acquainted with any new subject area, the pupil should engage in activities which are closely related to his existing knowledge and experience. He should not initially be faced with a heavy emphasis upon technical concepts and the relations between those concepts. Only after the learner has acquired sufficient experiential knowledge is it appropriate for him to work with a logically organized body of propositions within the subject area. In short, concrete experience precedes abstraction (1916).

2.6 GROWTH VERSUS PREPARATION AND TRAINING

Life and Growth

In everyday attitudes, growth is regarded as a major characteristic of life (1916, 1938). Adults are generally known to resist the notion that their growth is finished--that no further personal development is possible. Furthermore, adults are not usually regarded as basically deficient human beings. The adult possibility for development is viewed as a positive capacity, rather than as the lack of something which might be there at some time in the future (1916).

The same positive conception of growth is to be applied to the student (1916). The student should not be regarded comparatively as an incomplete, deficient, or imperfect adult, but as a living being with the capacity for full and rich experience at his present stage of development. "Since growth is the characteristic of life, education is all one with growing; it has no end beyond itself. The criterion of the value of school education is the extent in which it creates a desire for continued growth and supplies means for making the desire effective in fact" (1916, p. 53).

Preparation

The view of education as a process of continual growth stands in sharp contrast with the view of education as preparation for the future (1916, 1938). According to the latter conception, the learner is regarded as an incomplete adult, rather than as a being whose present existence has any intrinsic worth. As a candidate for adult life, his present stage of development is looked upon only as a means to an end. In preparation for the future, he is treated as an intellectual vacuum to be filled with knowledge and skills which may be of some future use. In the course of his preparation, the student is expected to maintain an attitude of "docility, receptivity, and obedience" (1938, p. 18).

It would be foolish, of course, to argue that schools should not prepare students for further education and for adult life. However, the conception of education solely as preparation for the future is based upon a dubious interpretation of the meaning of "preparation." There is another notion of preparation which is in accord with the view of education and of life as a continual process of growth (1916, 1938).

> What, then, is the true meaning of preparation in the educational scheme? In the first place, it means that a person, young or old, gets out of his present experience all that there is in it for him at the time in which he has it. When preparation is made the controlling end, then the potentialities of the present are sacrificed to a suppositious future. When this happens, the actual preparation for the future is missed or distorted. The idea of using the present simply to get ready for the future contradicts itself. . . . We always live at the time we live and not at some other time, and only by extracting at each present time the full meaning of each present experience are we prepared for doing the same thing in the future. [1938, p. 49]

Preparation and Training

The notion of education as preparation for the future is commonly associated with the classroom practice of training.* An animal can be trained by the application of the familiar carrot and stick method, under which the animal does whatever is necessary to get the carrot and avoid the stick. The student is often treated in a similar fashion; he is made to behave in a certain way, irrespective of his own preference, by means of rewards and punishments (1916). The particular action which he performs is of little concern to him, for his goal is simply to satisfy the teacher. He is not forming a disposition toward intelligent direction of his own activity, because the activity, from his viewpoint, possesses no intrinsic significance. He is being trained to act capriciously--to dispense with his own judgment and to simply do what he is told. Therefore, the student who undergoes training--even though he might acquire some information or skill as a result--is having a mis-educative experience (1916).

2.7 INTEREST, VALUES, SOCIAL CONTROL, AND FREEDOM

Interest

A person who is deeply interested in his work tends to concentrate, to be alert, to make careful observations, and to plan ahead (1916). A person who feels little interest in his work tends to daydream, rather than to devote his full attention to the task. Furthermore, when there is a genuine interest in the activity, the learner tends to persevere in the face of difficulty. When little interest is felt, the learner wishes to abandon the activity, rather than to strive to overcome the obstacle. Finally, under conditions of strong interest there is no need for the teacher to make threats or promises of irrelevant rewards, or to claim that the material might be good for something at some future date. There is also no need for moral exhortation--appeals to duty, self-discipline, or strenuous application of will. It is clear, therefore, that student interest is an essential aspect of the educative experience (1916).

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*In modern terminology, one form of training is known as "operant conditioning." Rewards and punishments are then called "positive and negative reinforcements" (Skinner 1953).

Values in Education

The interest of a person in a topic is closely related to his judgment of the worth, or value, of the topic. In making such a judgment, it is helpful to distinguish between intrinsic and instrumental values (1916). A person may value something intrinsically--for its own sake. On the other hand, the person may attach instrumental value to a topic which has no particular appeal in its own right. The topic is then valued as a means to an end; it is good for something.

The preceding brief discussion of values provides a criterion for the worth of educational experiences. "In general what is desirable is that a topic be presented in such a way that it either have an immediate value, and require no justification, or else be perceived to be a means of achieving something of intrinsic value" (1916, p. 242). What is not desirable is that a student perceive a particular topic to be of no value at all.

The emphasis in questions of value is on the student's response, not that of the teacher. It is definitely not sufficient for the teacher to say that a particular topic is interesting and important, for values are not communicated directly from one person to another. A person forms genuine standards of value, as opposed to merely verbal standards, by coming to realize the significance of things in specific situations (1916). For example, a teacher might state that it is important to define terms with precision and clarity, and the student might then offer lip-service to this standard. On the other hand, the student may be involved in some difficult, unclear situation, where precision and careful definition are necessary in order to achieve the desired goal. The student then forms his own standard of value, a working standard which can be applied with discrimination in varying situations (1916).

Social Control and Freedom

The importance of student interest and of the unforced formation of personal values sheds light upon the questions of control and freedom. It is often stated that students turned loose from the direct control of the teacher would do no work, and that the students would behave in an irresponsible manner, rather than paying attention to the task at hand. The statement ignores the role of interest, for students work energetically at tasks in which they feel a genuine interest (1916). There is also confusion as to the meaning of "control." When interest is felt in a shared activity, the control lies within the situation; the students have something which they want to accomplish and they go about doing so. " . . . The primary source of social control resides in the very nature of the work done as a social enterprise in which all individuals have an opportunity to contribute and to which all feel a responsibility" (1938, p. 56).

Control by joint participation in meaningful work is not personal or direct control on the part of the teacher (1916, 1938). The teacher does not find it necessary to "keep order," either by coercion or by moral exhortation; instead, the students acquire power to control their own activities. This type of situational control entails freedom from external restraint, which is a necessary condition for intelligent direction of one's own activity. Freedom from external restraint need not be regarded as an end in itself, but as a state which permits the development of independent thought and critical judgment, which is not possible when one's activity proceeds under the close direction of another (1938).

2.8 THE ROLE OF THE TEACHER

Neither freedom nor a genuine desire for further learning can be fostered by the teacher who forcibly imposes his will upon his students. However, the alternative to improper direction is not the withdrawal of the teacher from the learning situation (1916, 1938). In the first place, the teacher can become familiar with the interests and capabilities of the students; in the second place, the teacher's knowledge of subject matter is much greater than that of the students. To the teacher, " . . . the significance of a knowledge of subject matter, going far beyond the present knowledge of pupils, is to supply definite standards and to reveal to him the possibilities of the crude activities of the immature" (1916, p. 182). It is then possible for the teacher to furnish stimuli which channel the activity in a direction likely to be fruitful.

The two types of knowledge enable the teacher to see that certain experiences are of greater educational value than others; he can, accordingly, provide an environment in which it is likely that learning will occur. Once the environment is provided, the participation of the teacher does not stop, for the students are involved in some activity in which he can take part (1916, 1938). As a participant, the teacher should be concerned mainly with the response of the students to the subject matter, not simply with the subject matter itself. He must therefore possess both mastery of the subject matter and considerable perceptiveness and skill in dealing with other human beings. "When the parent or teacher has provided the conditions which stimulate thinking and has taken a sympathetic attitude toward the activities of the learner by entering into a common or conjoint experience, all has been done which a second party can do to instigate learning. The rest lies with the one directly concerned" (1916, p. 160).

2.9 SUMMARY OF A FRACTION OF THE DEWEYAN PHILOSOPHY

Education as an Active, Social, Personal Experience

1. Education is to be based upon personal experience of the learner.
2. An educative experience is agreeable in its own right and is connected with other such experiences in a cumulative manner. A mis-educative experience is one which hinders further development by producing, for example, mechanical skill without thought or by promoting indifference toward learning.
3. The individual learns by means of interaction with his environment; the teacher structures the learning environment, rather than teaching the student directly.
4. Interaction with the environment is an active process, rather than a matter of "passive absorption." The person actively does something, and in the course of his activity, he learns.
5. Learning in school is a social process in which the learner becomes a "partner in a shared activity." The activity takes place in a social group, as opposed to a collection of individuals.

Thinking, Nonthinking, Problem Solving, and Skill Formation

6. An educative experience must involve thought.
7. Thinking is not likely to occur in either the routine activity or the capricious activity, in which the individual follows the orders or directions of someone else.

8. Thinking may occur if the individual has a problem to solve, provided that the problem is interesting to the person and within his grasp, but difficult enough so that mistakes may occur.

9. Habits and skills should be formed under conditions where thought is necessary, rather than by means of routine and repetition. This can be accomplished by varying the environmental conditions during the period of skill formation.

Knowledge, Discovery Learning, and the Organization of Subject Matter

10. There is a distinction between knowledge, which arises in the course of purposeful activity, and the record of knowledge, the fruits of past activities.

11. Rather than simply absorbing the record of other people's activities, the student is to be involved in the discovery or creation of knowledge in the course of purposeful activities of his own.

12. Because of the tremendous gap in experience between the learner and the specialist in a particular discipline, the subject-matter organization preferred by the specialist may function as an obstacle to the student. Material should be organized in a form most conducive to learning, whether or not this form is logically satisfying to the specialist.

Growth Versus Preparation and Training

13. The student's possibility for further development is viewed as a positive capacity. The student is not to be regarded comparatively as an incomplete, deficient, or imperfect adult, but as a living being whose present existence has intrinsic worth.

14. Education is a continual process of growth which is worthwhile in its own terms, rather than solely a matter of preparation for the future. Preparation occurs properly as part of the process of growth by making the most of each present experience at the time when it occurs.

15. The view of education solely as preparation for the future is linked with the classroom practice of training by rewards and punishments, under which the student learns to dispense with his own judgment and to follow orders. Therefore, the student who undergoes training--even though he might acquire some information or skill as a result--is having a mis-educative experience.

Interest, Values, Social Control, and Freedom

16. The student who is genuinely interested in his work tends to concentrate, to be alert, to make careful observations, to plan ahead, and to persevere in the face of difficulty. Hence, student interest is an essential aspect of the educative experience.

17. A person may value something intrinsically, for its own sake, or instrumentally, as a means to an end which has intrinsic value. Every topic dealt with in school should have either intrinsic value or instrumental value from the student viewpoint.

18. " . . . The primary source of social control resides in the very nature of the work done as a social enterprise in which all individuals have an opportunity to contribute and to which all feel a responsibility" (1938, p. 56).

19. Situational control by joint participation in meaningful work entails freedom from external restraint by the teacher as a necessary condition for intelligent student direction of the activity.

Role of the Teacher

20. The teacher must know both his subject matter and his students. He does not forcibly impose his will upon the students, but he also does not withdraw from the learning situation. He provides an environment in which it is likely that learning will occur, and he participates in the activity of the students by providing sympathetic guidance at appropriate moments.

PART II: DESIGN OF THE SMALL GROUP-DISCOVERY METHOD

In the first part of this chapter, a description has been presented of the educational philosophy of John Dewey. On the basis of that philosophy extrapolated to the college level, a number of decisions were made in the design of the small group-discovery method. The major decisions involved the activity of the students, the role of the teacher, questions of interest and values, the presentation of subject matter, and the problem solving experience. In this chapter, only those decisions are reported which followed directly from the Deweyan philosophy; other decisions are reported in Chapter III.

2.10 DISCOVERY LEARNING IN SMALL GROUPS

Active Learning via Student Discovery

In the small group-discovery class, students were to learn mathematics by doing mathematics. Direct, active, mathematical experience was to replace the vicarious experience of listening and watching the lecturer do mathematics. The student was to be involved in the discovery of mathematical knowledge in the course of his own activity, rather than in the absorption of the record of other people's activities.

A particular item in the record of knowledge in calculus--a theorem, for example--might be of little intrinsic interest to the students. However, the attempt to formulate and prove the theorem might be highly exciting. The author believed that the record of knowledge in calculus could be largely re-created by the student themselves, once their attention was directed to certain areas of inquiry. The record would not be sacrificed in favor of student discovery, but would be re-generated in the process of student discovery.

In the small group-discovery class, the approach to the subject matter was to be one of guided discovery. Topics were often to be introduced by the teacher in the form of questions for investigation. The students would discuss these questions, and the answers might lead to the formulation of definitions or theorems. The proofs of the theorems were to be provided by the students, with hints given by the teacher when necessary. Techniques for solving various classes of problems were to be worked out by the students and not presented by the teacher.

Learning in Small Groups

In view of the Deweyan emphasis on shared activities, it was decided that students should work together in groups during the class period, rather than working alone. The work was to be done " . . . as a social enterprise in which all individuals have an opportunity to contribute and to which all feel a responsibility" (Dewey 1938, p. 56). The size of each work group could not be very large, since the opportunity for active participation and responsibility would decrease as the group size increased. Accordingly, each work group was to contain at most four members; the author believed that a work group with more than four members would contain a passive member. However, determination of the optimal group size appeared to be a matter for experimentation (see Chapter III).

The work groups were to have the following attributes. Within each group there was to be a cooperative atmosphere in which students worked together to solve the problem at hand. Group members were to provide answers to questions, which might be asked at any time without fear or embarrassment. Every member would understand each concept or problem solution before any new work was done. The students would be free to determine both the membership of the work groups and the time schedule for changing membership. The description of the work groups is considerably expanded in Chapter III.

2.11 THE ROLE OF THE TEACHER

Structuring the Environment

The teacher was not to teach the students directly, but to structure the environment for the facilitation of learning. The physical environment was to consist of a room with a large blackboard and an ample supply of chalk and erasers for the students. The social environment was to consist of the entire class, including the teacher, the shared activities taking place within the small group setting, a non-threatening social atmosphere, and a set of norms, roles, and policies to be described later in this chapter and in the succeeding one.

The mathematical environment was to consist of the subject matter of calculus, including concepts, definitions, theorems, proofs, examples, counterexamples, and various classes of problems. The teacher was to structure the mathematical environment mainly by raising questions for investigation. For example: How could we find the tangent line at a point on an arbitrary curve in the plane? What happens at a high or low point on a curve? How can we compute the area under a given curve?

Leadership Style and Functions

In addition to providing an environment in which learning was likely to occur, the teacher was to become a participant in the activities of the students. The teacher would have the choice of forcibly imposing his will on the students (the autocrat) or of participating to the smallest possible extent (the laissez-faire leader). Each of these alternatives was rejected in favor of the democratic leadership style in which the teacher was to participate actively but not in an overly directive manner. The leadership style of the teacher is described in detail in Chapter III.

The teacher was to spend most of the class period with the small work groups. He was to keep track of the progress of the groups, make corrections and suggestions, clarify notation, give hints, check solutions, provide encouragement, and try to see that the groups functioned smoothly. He might talk with the entire class at different times during the period, generally for no more than five or ten minutes per day. During the brief discussions with the entire class, he was to present new concepts, raise questions, answer certain questions, assign problems, give hints, clarify and summarize what the students had done, and serve as discussion moderator. Each day he was to prepare and distribute notes containing a record of the students' accomplishments; the written notes were intended to take the place of a textbook.

Attitudes toward the Students

In accordance with the Deweyan philosophy, the small group-discovery class was to be based upon the view of education as a process of continual growth and not upon the view of education as preparation for the future, with the accompanying practice of training. This decision had several practical implications.

The teacher would not treat the students as deficient, incomplete, imperfect adults, who were in college solely to be prepared for the future and who were too ignorant or inexperienced to know their own interests and needs. When students questioned the value of a particular topic, he would not state that the topic was necessary preparation for future mathematics course work, next semester, next year, or in graduate school. He would not practice training via the rewards and punishments of the grading system, exemplified by statements such as: "If you don't learn it, you'll flunk the test; if you do learn it, you might get a good grade." Finally, he would not practice training with verbal reinforcement techniques, such as: "Yes, that's a very good solution," or "No, you're way off base."

The teacher was to behave in a manner consistent with the view of education as a continual process of growth. He would show considerable respect for the students as individuals, as "living, experiencing beings." He would recognize the rights of students to be interested or not interested in a given mathematical topic. He would work in accordance with present student interests and attempt to create further interest along fruitful lines of inquiry. He would have enough trust in the students to let them discover most of the mathematics themselves, rather than to tell it all himself. He would realize that experience which "creates a desire for continued growth" is not something to be forced upon students.

2.12 STUDENT INTEREST AND THE FORMATION OF VALUES

Interest versus Grades

Emphasis upon grades was to be substantially reduced, if not eliminated, in the small group-discovery class. Interest in the mathematics was intended to provide the major source of motivation. In order to find out what topics were interesting to the students, it was necessary to remove the artificial pressures and inducements to learn things that were not of interest. Since students would feel most free to express their opinions in a non-threatening environment, it was desirable to greatly diminish the grading power of the teacher.

Several alternatives were considered for grading in the small group-discovery class. The most preferable alternative was the elimination of grading, but this was prohibited by university regulations. The next alternative was an automatic passing mark in a pass-fail system. However, freshmen were not allowed to take the pass-fail option, and a letter grade had to be given.

The third alternative was an automatic A grade, guaranteed in advance to every student in the class. This alternative was seriously considered but finally rejected for the following three reasons:

1. Students would be graded in all their other courses and, when the time arrived to take tests and write papers elsewhere, they would be tempted to put these priority matters before their mathematics.

2. The students had been graded for many years. If the grade were removed, there might be a period of release behavior during which certain students would choose to do no mathematics.

3. The class might include certain students who were used to working only for the grade. If the grade were guaranteed to be an A, then no work would be done by these students, no matter what mathematics was presented for their enjoyment.

The three reasons indicated that an isolated grade-free class in a grade-filled environment might encounter serious difficulties. It was then reluctantly decided to use an A-B grading scale in the small group-discovery class; this decision did not live up to the ideals of the Deweyan philosophy. However, it did appear that the A-B scale would greatly restrict the amount of extrinsic motivation. The grading pressure was to be further reduced by eliminating examinations in class and by leaving further determination of grading policies up to the students.

Standards of Value Regarding Content

In a truly educative environment, every mathematical topic would possess either intrinsic value or instrumental value for the students. One of the basic goals in the small group-discovery method was to determine which topics had intrinsic value, which had instrumental value, and which had no value from the student viewpoint. There was the possibility, of course, that different students might form different standards of value about the calculus content.

If a topic appeared to possess no value from the student viewpoint, the teacher might be able to ascertain a reason for the lack of value. It is possible that the topic did not connect with the student's previous experience, that a connection existed but was not clear, that the topic appeared to be useless, that the level of difficulty was too high or too low, or that the learning activity was being conducted in a routine or capricious manner. In none of these cases was the student to be blamed for his lack of enthusiasm. Instead, if there was some good reason for retaining the topic, its placement or presentation could be changed to remedy the motivational difficulty. However, if various changes did not increase the student evaluation of the topic, there would be a serious question about the suitability of that topic for inclusion in a calculus course.

It was expected that students with increased problem-solving experience would become more readily able to form their own standards of value about the mathematics content. It would become unnecessary for the teacher to proclaim, "This is an important theorem," for a result which was useful in the solution of other problems would tend to acquire high instrumental value. With repeated use, what was once of instrumental value might even take on intrinsic value.

Standards of Value Regarding Rigor

If the teacher placed high value on rigor, he might be tempted to impose his own standards of mathematical rigor upon the students; there would be dangers in such an imposition. To the students, the standards of the mathematician might appear to be unreasonable and arbitrary. The mathematician might appear to be engaged in mere quibbling, or he might be laboring to clarify some point that needed no clarification as seen by the students, with the possible result of obscuring what was already understood.

There was also the danger of student attempts to adopt a standard of rigor beyond their own level. For example, it is well known that professionally written, perfected mathematical proofs have sometimes disguised the central ideas in the solution of the problem. Attempts by students to imitate the perfected professional form might result largely in confusion and poor communication. Moreover, the students might become so enmeshed in the formalism as to lose sight of the basic ideas under consideration. The ability to discover results and bring forth new ideas might be seriously reduced by anxiety over the expression of the ideas in the most impeccable form. In brief, if all ideas had to be impeccably expressed, there might be very few ideas left for expression.

There were two possible dangers, then, with respect to professional standards of mathematical rigor. In the first place, the students might resent the imposition of seemingly unreasonable and arbitrary standards. In the second place, the students might accept and attempt to copy standards beyond their own level, with the possible outcomes of confusion, poor communication, and inhibition of thought.

It appeared that the previously cited dangers could be minimized by helping the students to form their own standards of rigor. As a general guideline, there was to be as much rigor and clarity as demanded by the situation itself. For example, if one student could not understand the vaguely expressed ideas of another, then greater clarity would be necessary. Perhaps the use of certain terminology would permit the statement of a difficult point in a lucid manner. In addition, a key point might be passed over during the discussion, or a false assumption might be made. In each of these instances, more careful thought and expression would be necessary for the problem solution; however, this would not be a demand for rigor for its own sake.

In conclusion, a high standard of rigor might have intrinsic value for the teacher, but for students of calculus the same standard might not even possess instrumental value. Nevertheless, it was expected that the students, in handling increasingly difficult and abstract problems over the course of time, would themselves see the need for increased precision and theoretical security. This unforced growth with respect to standards of rigor was expected to accompany the formation of personal values regarding the mathematics content, as previously described.

2.13 THE PRESENTATION OF SUBJECT MATTER

Organization of Subject Matter

The subject matter for the calculus class was to be organized in a form most conducive to learning, irrespective of the criterion of logical elegance. The presentation of subject matter was planned to proceed from the more concrete to the more abstract, as seen from the viewpoint of the learner. "Concrete" and "abstract" were interpreted as follows: Subject matter which is closely related to the existing knowledge and experience of the learner appears to be concrete. Subject matter which is remote from the existing knowledge and experience of the learner appears to be abstract.

An example of the development from concrete experience to abstraction can be provided by a unit dealing with limits of sequences. The unit was to begin with a student discussion of the arrow paradox and the problem of finding the area of a circle. After these topics, several particular sequences could be introduced for the students' consideration. Questions would be raised: What is the point of these examples? Are there any common properties among these sequences? Are there any major differences? How can you describe this phenomenon (either in words or nonverbally)? Can you construct any more examples of this type?

There would follow a discussion of the meaning of convergence and divergence. A number of sequences were to be classified as either convergent or divergent. Certain definite integrals representing areas were to be computed as limits of sequences, and the number "e" was to be approximated. Sequences were to be combined in various ways, and the convergence or divergence of these combinations and of certain infinite series were to be investigated.

After consideration of numerous particular examples, certain puzzling sequences were to be discussed. Hopefully, the students would realize that the convergence or divergence of these sequences could not be ascertained without a precise definition for the limit of a sequence. The students might then be able, with limited guidance, to formulate the definition of the limit by considering common properties of various convergent sequences. The specific details are presented in Chapter V.

The unit dealing with limits of sequences was to be inserted at the very beginning of the small group-discovery class. The author believed that the concept of the limit of a sequence is much easier to work with than that of the limit of a function of a continuous variable. In comparison of the two types of limits, it would seem less confusing for the students to formulate the definition of a limit in the sequence case and to later prove some of the basic limit theorems. There might then be considerable transfer of learning to the continuous variable situation.

Skills with Classes of Problems

The subject of calculus includes many classes of problems. There are, for example, problems involving

(1) computation of limits, derivatives, and integrals, (2) curve sketching, applied maxima and minima, and velocities, (3) areas, volumes, and arc lengths. In each class of problems, skill could be attained by solving a number of specific problems of the general type. Hence, there were the dangers that skill formation might take place in either a routine, mechanical fashion or in a capricious fashion, if the students were made to solve problems of little interest.

In accordance with the Deweyan philosophy, it was decided that skills should be formed under conditions where thought was necessary--that students should not be presented with a number of routine problems to be solved. In the small group-discovery class, the students themselves were to develop the techniques for solving each class of problems. The remaining practice could then occur with problems which differed from one another and which required some judgment in finding the solutions.

It was decided that skill-building could not take place in a capricious fashion if the skills were attained by solving problems of intrinsic value for the students. For example, practice in differentiation could be attained by solving problems of type (2)--curve sketching, maxima and minima, and velocities. Practice in integration could be gotten by solving problems of type (3)--areas, volumes, and arc lengths. It was assumed that certain problems of type (2) and of type (3) would be intrinsically interesting to the students.

2.14 THE PROBLEM SOLVING EXPERIENCE

The Nature of Problem Solving

The author's view of problem-solving for college students in calculus coincided partially with the Deweyan view of problem-solving. For Dewey, it was " . . . indispensable to discriminate between geuine and simulated or mock problems. . . . Is it the pupil's own problem, or is it the teacher's or textbook's problem. . . ." (Dewey 1916, p. 155). Moreover, in the Deweyan view, the student was to pursue some ongoing, purposeful activity, in which he was intrinsically interested. The problem would then arise as a difficulty, or obstacle, in the course of this activity, and the obstacle would have to be removed or circumvented in order for the activity to proceed. Preferably, the activity would be of the type that would naturally interest the student outside of the school situation, so that there would be no division between the vital interests of life and the more remote and abstract subject matter of the school.

Problem-solving in mathematics and other courses usually has a much different meaning than that described above. Arnstine has made this point succinctly, in his analysis of proposals for problem solving. " . . . These current discussions do refer to a sort of problem-solving, but the problems usually turn out to be ones invented by teachers and textbook writers that are simply presented to students. The students are then told to solve them" (Arnstine 1967, p. 152).

Arnstine goes on to discuss the inherent limitations of the Deweyan view of problem solving; some of these limitations are presented here in the context of planning the small group-discovery class. In the first place, the students could be expected to have a wide range of interests and purposes; it might be quite difficult to find several students with the same mathematical purpose. Even if it were possible to form several groups of students, each group consisting of students with the same purpose, it might be the case that certain students would have the background knowledge to attack the problems which developed, while others would not. Moreover, problem-solving activities could easily be selected which were far too difficult at the learners' particular stage of development. In addition, the teacher would have the confusing responsibility for supervision of several essentially different kinds of problem-solving activities during the same class period. Finally, the activities selected from out-of-school experience might be so diverse as to require of the teacher a practically impossible range of knowledge (Arnstine 1967).

In light of the previous discussion, it was decided to abandon the criterion that problems should always arise in the course of an ongoing, student-selected activity which would be of interest to students outside of the university context. Nevertheless, it was possible to meet Dewey half-way: Dewey was concerned that a problem would be " . . . made a problem for the pupil only because he cannot get the required mark or be promoted or win the teacher's approval unless he deals with it. . . ." (Dewey 1916, p. 155). As previously mentioned, the author was to seriously attempt to determine which problems had intrinsic value, which had instrumental value, and which had little value to the students. The students would be free to be interested or not interested in a given problem. No one was to be forced, and problems which met with strong objections were to be discarded.

In meeting Dewey half-way, the teacher could present a number of mathematical problems in the usual sense which he believed would be enjoyable to students. Moreover, within the basic mathematical framework provided by the teacher, it was expected that many questions would occur to students. The investigation of student-generated questions was intended as one of the important activities of the small group-discovery class. It was hoped that within this exploratory type of activity, problems in the Deweyan sense would arise and be resolved by the students.

Avoidance of Mis-educative Experiences

It appeared that the small group-discovery method would include a check against the occurrence of mis-educative experiences. It was anticipated that the teacher, through his close contact with the students, would receive a good deal of immediate feedback. The teacher could then tell whether the students were actively involved and interested or bored, indifferent, or frustrated. He could tell when the students were operating in a mechanical fashion, and he could quickly identify errors and misconceptions. In addition, he would probably know which points were adequately grasped. The immediate feedback would allow the teacher to alter the learning situation if it appeared that the students were not having a fruitful experience.

2.15 SUMMARY

The pilot study for the small group-discovery method was to be conducted for one academic year with a freshman calculus class. The major instructional practices derived from the Deweyan philosophy are summarized as follows:

1. Students were to learn mathematics by doing mathematics. The approach to the subject matter was to be one of guided discovery in which topics would be introduced in the form of questions for investigation by the students. The students, with some guidance from the teacher, were to formulate definitions, state theorems, prove the theorems, construct examples and counterexamples, and develop techniques for solving various classes of problems.

2. The classroom activity was to be a social process taking place in small groups of students, with no more than four members in each group. Within each group there was to be a cooperative atmosphere in which students worked together to solve the problems at hand.

3. The teacher was to adopt a democratic leadership style; he would become a participant in the activities of the students, but not in an overly directive manner.

   The teacher was to spend most of the period with the small work groups. He was to keep track of the progress of the groups, make corrections and suggestions, clarify notation, give hints, check solutions, provide encouragement, and try to see to it that the groups functioned smoothly. He might talk with the entire class at different times during the period, generally for no more than five or ten minutes per day. During the brief discussions with the entire class, he was to present new concepts, raise questions, answer certain questions, assign problems, give hints, clarify and summarize what the students had done, and serve as discussion moderator.

4. There was to be no textbook for the course. The textbook was to be replaced by daily notes prepared by the teacher and containing a record of the students' accomplishments.

5. Interest in the mathematics was intended to provide the major source of motivation, and there was to be a non-threatening classroom atmosphere in which students could feel free to express their opinions. Emphasis upon grades was to be substantially reduced by an A-B grading scale, by the elimination of examinations in class, and by student determination of further grading policies.

6. The teacher would attempt to determine which topics were of intrinsic value, which appeared to be useful (instrumental), and which had no value from the student viewpoint. The intent was to provide a learning environment in which every topic possessed either intrinsic value or instrumental value for the students.

7. Emphasis was to be placed upon the discovery of new ideas, rather than upon the expression of the ideas in the most impeccable form. Professional standards of rigor were not to be imposed upon the beginner, and the initial development was to be informal in character. The need for increased precision and theoretical security was expected to become apparent to the students with the handling of increasingly difficult and abstract problems over the course of time.

8. The presentation of subject matter was to proceed from the more concrete to the more abstract, as seen from

the viewpoint of the learner. Abstract, theoretical considerations were to be postponed, pending the occurrence of a good deal of concrete experience.

9. Skills were to be formed under conditions where thought was necessary. The students themselves were to develop the techniques for solving each class of problems. The remaining practice would occur with problems which differed from one another and which required some judgment for the solutions. The skills were to be attained, whenever possible, by solving problems of intrinsic value for the students.

10. The teacher was to propose problems, in order to guarantee that all major topics of the calculus were covered during the course. Within the basic framework provided by the teacher, it was expected that many questions would occur to the students. The investigation of student-generated questions was intended as one of the important activities of the class members.

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Chapter III

GROUP DYNAMICS AND THE SMALL GROUP-DISCOVERY METHOD

3.1 OUTLINE OF CHAPTER III

In the previous chapter, a number of decisions have been reported regarding the design of the small group-discovery method on the basis of the Deweyan philosophy. The purpose of the present chapter is to provide empirical support for some of these decisions and to elaborate upon the existing design, placing emphasis upon the classroom social climate. The present chapter includes a review of a number of studies in the field of group dynamics and a report of some additional decisions grounded upon those studies. The major variables under consideration in this chapter are the following: (1) the leadership style of the teacher, (2) the issue of cooperation or competition in a work group, (3) conformity pressure, (4) leadership by students, (5) group size, and (6) grading policies.

3.2 EXPERIMENTAL STUDIES RELATED TO PREVIOUS DECISIONS

There now follows a brief summary of previously reported decisions concerning the six major variables and a brief description of the type of experimental evidence to be cited for each variable.

1. On the basis of the Deweyan philosophy, a particular leadership style was prescribed for the teacher of the small group-discovery class. The teacher was not to impose his will upon the students, nor was he to withdraw from the learning situation. Instead, he was to structure the environment to promote learning and to participate with a "sympathetic attitude" in the activities of the students. This leadership style, in order to be useful, needed to be described in operational terms. In the present chapter, studies are cited which specifically describe the leadership style and which provide empirical support for it.

2. In keeping with the Deweyan philosophy, there was to be a cooperative atmosphere within each work group of the small group-discovery class. In the present chapter, there is an empirically based description of the effects of cooperation and of competition in problem solving groups, and a discussion of ways to promote cooperation.

3. In accordance with the views of Dewey, emphasis in the calculus class was to be placed upon independent thinking and the unforced formation of personal values. These outcomes would not be readily attainable in a group where there was conformity pressure. In the present chapter, consideration is given to investigations of the effects of conformity pressure and to studies which indicate means of reducing conformity pressure.

4. The work in the small group-discovery class was to be done " . . . as a social enterprise in which all individuals have an opportunity to contribute and to which all feel a responsibility" (Dewey 1938, p. 56). The question arose as to whether all individuals were to contribute the same amount and to feel equally responsible for the outcome of the group activity. In particular, should there be a designated leader for each work group? The present chapter includes some empirical evidence about power relations and leadership in task groups.

5. In accordance with the Deweyan emphasis upon learning through active participation in problem solving, it was decided to restrict the size of each work group to a small number of members--say no more than four members. In this chapter some experimental evidence is cited on the

functioning of discussion groups having varying numbers of members.

6. In order to minimize anxiety, to reduce the impact of the "extraneous inducements" of the grading system, and to allow the students to be motivated by interest in their work, the decision was made to use an A-B grading scale and not to give examinations in class. Further specification of the grading system was to be determined by the students. In this chapter there is an empirically based discussion of the effects of grades and anxiety upon learning and problem-solving performance.

3.3 THE LEADERSHIP STYLE OF THE TEACHER

In the planning of the small group-discovery class, a number of questions arose concerning the role of the teacher. What should be the teacher's behaviors in such a class? How much of his authority should he use, and when and how should he use it? What type of interpersonal relationship should prevail between the teacher and the class members? What functions should the teacher perform in the supervision of the work groups? Answers to these questions were suggested by the research of social psychologists in the area of leadership.

Three different styles of leadership have been described and experimentally studied by White and Lippitt (1960) under the general direction of Kurt Lewin. The three basic leadership styles were named "authoritarian, democratic, and laissez-faire." The work of Lewin, Lippitt, and White has in turn stimulated a number of other investigations in the area of leadership--some of which have dealt with college students in organized academic classes. The results of these other investigations are reported in this paper after a description of the study by White and Lippitt.

Authoritarian (Autocratic), Democratic, and Laissez-Faire Leadership

In the classic study conducted by White and Lippitt (1960), the adult leaders of boys clubs varied their leadership styles. Each of the leaders was trained to proficiency in the authoritarian (autocratic), democratic, and laissez-faire styles of leadership. The experiment was conducted with five-member groups of ten-year-old boys meeting after school in hobby activities. Each group of boys experienced each of the three leadership styles for a period of six weeks, and the effects upon group life and individual behavior were recorded. The chief characteristics of each style of leadership have been summarized in a table prepared by White and Lippitt (1960, p. 528). (See p. 109.)

Behavior of the Leaders in the White and Lippitt Study

The characteristics of the three leadership styles presented in Table 3.1 included some but not all of the differences between the three styles. Other differences appeared during the meetings of the boys' clubs, during which the behaviors of the leaders were observed and sorted into various categories. In the eight categories which follow, there were statistically significant differences in the behaviors of the three types of leaders. The numbers represent percentages of the leaders' total number of behaviors; the percentages are included here only to assist the reader in distinguishing between the various styles.

The first 4 categories were more characteristic of the autocratic leaders than of the democratic or laissez-faire leaders.

1. Giving Orders

Autocratic 45%	Democratic 3%	Laissez-faire 4%

The largest single statistical differentiation between the autocratic style and the other two styles was the giving of orders.

TABLE 3.1

CHARACTERISTICS OF THE AUTHORITARIAN, DEMOCRATIC, AND LAISSEZ-FAIRE STYLES OF LEADERSHIP

	Authoritarian	Democratic	Laissez-faire
1.	All determination of policy by the leader	All policies a matter of group discussion and decision, encouraged and assisted by the leader	Complete freedom for group or individual decision, with a minimum of leader participation
2.	Techniques and activity steps dictated by the authority, one at a time, so that future steps were always uncertain to a large degree	Activity perspective gained during discussion period. General steps to group goal sketched, and when technical advice was needed, the leader suggested two or more alternative procedures from which choice could be made	Various materials supplied by the leader, who made it clear that he would supply information when asked. He took no other part in work discussion
3.	The leader usually dictated the particular work task and work companion of each member	The members were free to work with whomever they chose, and the division of tasks was left up to the group	Complete nonparticipation of the leader
4.	The dominator tended to be "personal" in his praise and criticism of the work of each member; remained aloof from active group participation except when demonstrating	The leader was "objective" or "fact-minded" in his praise and criticism, and tried to be a regular group member in spirit without doing too much of the work	Infrequent spontaneous comments on member activities unless questioned, and no attempt to appraise or regulate the course of events

2. Disrupting Commands

Autocratic 11% Democratic and Laissez-faire 1% or less

Disrupting commands "cut across an expressed wish or ongoing activity of a member of the group, and substitute for it some wish of the leader" (White and Lippitt 1960, p. 531).

3. Non-Objective Criticism

Autocratic 5% Democratic 1% Laissez-faire 1%

Non-objective criticism was defined as "criticism which was adverse and personal in character and which did not point objectively toward improvement by suggesting a reason for failure or a way of doing the thing better" (1960, p. 531).

4. Non-Objective Praise

Autocratic 11% Democratic 7% Laissez-faire 1%

The large amount of personal evaluation--both criticism and praise--by the autocratic leaders suggests the existence of a status hierarchy in which the autocrat determined the standing of the group members.

The following categories were more characteristic of the democratic or the laissez-faire leaders than of the autocratic leaders.

5. Guiding Suggestions

Democratic 24% Laissez-faire 14% Autocratic 6%

Guiding suggestions were not orders or commands but were related to the member's own purpose. The suggestions were to be given only when they would be needed and appreciated. The main characteristic which distinguished the democratic from the laissez-faire leaders was the giving of guiding suggestions. The democratic leaders were much more ready to give suggestions at the appropriate times and to explain the rationale for the suggestions.

6. Giving Information

Laissez-faire 49%   Democratic 27%   Autocratic 15%

Giving information was the same as extending knowledge, as opposed to making suggestions about a course of action. The major activity of the laissez-faire leaders was the giving of technical information upon request. The other leaders actually provided about as much information as the laissez-faire leaders but spent a higher proportion of their time in other kinds of leader behaviors.

7. Stimulating Self-Direction

Democratic 16%   Laissez-faire 13%   Autocratic 1.2%

Although the first two percentages were fairly close, the democratic leaders averaged about twice as many such acts per meeting. The democratic leaders stimulated self-direction in two ways. They encouraged the use of open discussion and the democratic process in making decisions about group goals and means. On the individual level, they encouraged members to elaborate and think through their ideas.

8. Jovial and Confiding Behavior

Democratic 8% Autocratic and Laissez-faire less than 1%

The democratic leaders, with their willingness to discuss non-business matters, developed a social relationship with the group members.

9. Equalitarian Behavior

This category was not statistically analyzed, but it did contribute to the total impression made by the leader. The democratic leaders showed little concern about their status and dignity. They took off their coats, sat with the boys instead of standing, worked along with the boys, and showed enthusiasm for the task. Their behavior was characterized as equalitarian and self-effacing in comparison with the other types of leaders.

Major Differences in Boys' Behavior in the White and Lippitt Study

1. Democracy versus Laissez-Faire

The observations showed that more work was done in the democratic situation and the quality of work was higher than in the laissez-faire situation. The boys in the laissez-faire groups spent more time playing, and when the laissez-faire boys did work, there were often setbacks and disorganization. In interviews the boys reported a preference for the democratic leader.

2. Work Efficiency in Democracy versus Autocracy

It is necessary to distinguish between submissive and aggressive reactions to autocracy. In those autocratic groups in which the boys behaved submissively toward the leader, more time was spent working than in the democratic groups. In the autocratic group which reacted aggressively, about the same amount of time was spent working as in the democratic groups. "On the other hand, the amount of genuine interest in work was unquestionably higher in democracy" (1960, p. 540). There was more work-minded conversation in democracy, and there was only a very slight drop in work involvement when the leader left the room. In both the submissive and the aggressive reactions to autocracy, there was a tremendous drop in work involvement when the leader left the room. Furthermore, the democratic groups displayed more originality and creative thinking than the autocratic groups, according to the impression of the experimenters.

3. Hostility and Aggression in the Aggressive Reaction to Autocracy

The following behaviors were conspicuous in the aggressive (but not in the submissive) reaction to autocracy: domination of one person over another, overt hostility, aggressive demands for attention, destruction of one's own property, and scapegoating. These behaviors either were rare or else did not occur at all in the democratic situation; no comparison was presented with the laissez-faire situation.

4. Hidden Discontent in the Autocratic Situation

In each type of autocratic group, there were more expressions of discontent than in the democratic groups. In addition to the overt expressions of discontent, there were other manifestations of dissatisfaction with the autocratic leaders. (1) Several boys dropped out of the clubs with autocratic leaders. All the dropouts occurred during times when there was no open rebellion against the autocrat. (2) In confidential interviews, the boys almost unanimously chose the democratic over the autocratic leaders. (3) There was often "release" behavior on the day of transition from autocracy to another leadership style. The boys aggressively "blew off steam" on the transition day.

5. More Dependence and Less Individuality in Autocracy

In the autocratic groups, as compared with the democratic groups, the boys showed many more submissive and dependent actions toward the adult leader. The conversation included less variety and was more restricted to the immediate setting. In the submissive reaction to autocracy, the atmosphere was subdued, sober, and low-spirited.

6. Group-mindedness and Friendliness in Democracy

Spontaneously formed subgroups were larger in the democratic groups than in the autocratic groups. The autocratic groups tended to disintegrate frequently, despite efforts by the leader to promote more unity. There were many more group-minded remarks in the democratic situation than in the other leadership situations. There was slightly more friendliness in democracy than in autocracy.

The results of the White and Lippitt study seemed to indicate the desirability of the democratic leadership style for the teacher in the small group-discovery class. However, it was not evident that the results obtained with ten-year-old boys in hobby activities would also hold for college students in organized academic classes. It was therefore appropriate to examine some further evidence with respect to classroom styles of leadership.

A Review of the Authoritarian-Democratic Studies

Anderson (1963) has presented a review of forty-nine experimental studies in which leadership was defined by a "hypothetical authoritarian-democratic dimension." Although the idea has been "basically the same" in all the studies, a number of different names have been used for the dimension in question. The names have included the following: authoritarian--democratic, teacher-centered--learner-centered, dominative--integrative, directive--non-directive, supervisory--participatory, therapist-centered--client-centered, employer-centered--employee-centered.

Anderson has presented a "rough working definition" of the authoritarian and democratic leadership styles. "Authoritarian leaders generally act in an impersonal manner; punish members who disobey or deviate; decide what the group will do or talk about; decide upon division of labor; determine how work shall be done; judge the soundness of ideas; do more talking than the other members of the group. Democratic leaders generally act in a friendly, personal manner; allow the group as a whole to plan the agenda; allow group members to choose the tasks they will perform; allow group members to talk with one another without permission from the leader; accept suggestions from the group about how the work shall be done; talk only a little more than the average group member. In short, the authoritarian leader makes most decisions for the group whereas the democratic leader shares decision-making with the other members" (Anderson 1963, p. 154).

Thirty-two of the forty-nine studies reviewed by Anderson have dealt with styles of leadership in educational settings; the results have not been conclusive with respect to measures of student learning. Eleven studies have reported more learning in learner-centered classes, thirteen studies have reported no difference, and eight studies have reported more learning in teacher-centered classes. However, with respect to morale, "it is a rather general finding that morale is higher in learner-centered groups. . . . The only exceptions to this generalization appear to occur in situations in which there is high anxiety about grades which are awarded on the basis of final examination scores. . ." (Anderson 1963, p. 158). Morale was defined by Anderson as "the extent to which group members find the group personally satisfying and the extent to which they believe the group is successfully progressing toward its goals" (1963, p. 154).

Anderson has mentioned the need for caution in interpreting the results of the various investigations. In the studies conducted with college classes, for example, the outcomes defined as "learning" have been measured in diverse ways, ranging from multiple-choice examinations to counts of the number of insightful comments made by students. There have been similar discrepancies in the operational definitions of the term "morale." Moreover, among the numerous investigations, the operational definitions of the democratic and authoritarian leadership styles have varied considerably. Finally, the authoritarian leaders in some of the investigations have been "unreasonably harsh and austere"; different results might occur with benevolent autocrats who behaved in a more friendly and personal manner.

Decisions for the Pilot Study

On the basis of the available evidence, the author made some decisions about the leadership style of the teacher of the small group-discovery class. The decisions were based largely upon the work of White and Lippitt and were consistent with the findings reported in Anderson's review of authoritarian-democratic studies.

The author rejected the laissez-faire style of leadership for the small group-discovery class because more work and work of higher quality could be accomplished in the democratic situation (White and Lippitt 1960). It should be noted that in the studies reviewed by Anderson (1963)--studies which were greatly affected by the previous work of Lewin, Lippitt, and White--the laissez-faire style was in most instances not even included as a possibility for investigation.

It was then necessary to choose between the democratic and the authoritarian styles. The articles by White and Lippitt and by Anderson did not conclusively show that more work or more learning could be expected in either the democratic or in the authoritarian situation. Hence, the decision had to be made on other grounds, namely upon considerations of morale and of interest in the task.

According to Anderson, morale has been generally higher in democratic classroom groups than in autocratic classroom groups. Some of the specific indicators of low morale can be identified by again returning to the results of White and Lippitt. One recalls that the autocratic groups were characterized by hidden discontent, and that the tendency toward group fragmentation indicated a lack of common effort on the task. In one reaction to autocracy, there was a large amount of hostility and aggression; in the submissive reaction to autocracy, the group atmosphere was subdued and low-spirited. Finally, in both reactions there were submissive and dependent actions toward the adult leader.

Interest in the task was to be the major motivational force in the small group-discovery method. According to White and Lippitt, genuine, unforced work interest was higher in democracy than in autocracy. This finding was confirmed in several studies reported by Anderson, including one study conducted by himself.

On the basis of the evidence with respect to morale and interest in the task, the author decided to use the democratic style of leadership in the small group-discovery class. This decision was consistent with the Deweyan philosophy described in Chapter II. According to Dewey, the teacher should neither impose his will upon the students (the autocrat) nor withdraw from the learning situation (the laissez-faire leader). He should instead adopt an intermediate style of leadership by structuring the environment to promote learning and by participating with a "sympathetic attitude" in the activities of the students. The democratic leadership style described in detail by White and Lippitt could then be regarded as an operational definition of the teaching style prescribed by Dewey.

It remained to specify the particular way in which the democratic leadership style was to be applied in the small group-discovery class. This was done in accordance with the definitions presented in Table 3.1 and with the other leader behaviors observed experimentally by White and Lippitt. The behavior of the teacher is discussed here in terms of the following general guidelines; the application of the guidelines in practice is described in Chapter IV.

1. Most policies in the small group-discovery class were to be arrived at through group discussion and decision-making by a majority vote. The teacher was to serve as moderator of the group discussions. Some of the policies to be discussed involved matters such as the method of grading, the manner of formation of the work groups, and standards of behavior for the work groups.

2. A perspective on each day's mathematical activities was to be provided by the teacher at the beginning of each class period in a discussion with the entire class. These discussions would generally last for no more than five or ten minutes per day, during which time the teacher was to present new concepts, raise questions, answer certain

questions, assign problems, give hints if necessary, and serve as discussion moderator. There would sometimes be brief discussions at the end of the class period, during which time the teacher might clarify and summarize the students' accomplishments, answer questions, and set the stage for the next day's work.

3. During the first few weeks of the class, the membership of the work groups was to be changed every two or three days, so that each member of the class could become acquainted with all the other members. After the initial period of acquaintance, the students would be free to decide upon the membership of the several work groups. Within each group the students were to work together to solve the problems at hand but would otherwise be free to determine the manner of procedure and division of labor.

4. The teacher was to give only a minimal amount of objective, constructive praise and criticism, usually directed to the group as a whole, rather than to particular individuals.

5. The teacher was to refrain from giving either orders or disrupting commands.

6. The work was to be done by the students in their small groups, with help available from the teacher as needed.

The teacher was not always to wait for a request for assistance, but to keep track of the progress of the groups and to offer suggestions at times when they would be needed and presumably appreciated.

7. Guiding suggestions of a mathematical nature were to be given in the form of hints, sometimes using the heuristic techniques of Polya.

8. The teacher would sometimes offer guiding suggestions with respect to the organization of the work within a particular group and with respect to the functioning of the group as a social unit. For example, if the students were writing all over the blackboard in a confusing, disorganized manner, the teacher might say the following: "You might make faster progress if you wrote things down systematically and erased the ideas which didn't work." If one person were dominating the group and doing most of the problem-solving himself, the teacher could suggest a system of rotation in which different members were responsible in turn for writing solutions on the blackboard. This suggestion could be accompanied by an explanation that the point of the class was for everyone to be actively involved in the problem solving.

9. The teacher was to provide certain types of

technical information upon request. For example, he might tell the students a particular trig identity, if the development or recall of the identity was not judged to be an important aspect of the problem at hand. However, upon occasion the teacher might state, "That formula comes up very often, and you should be able to develop it or remember it whenever you need it."

10. The teacher was to stimulate self-direction by encouraging individual members to think through and elaborate upon their ideas by explaining them to the other group members. At times the teacher was to stimulate self-direction by simply stating that there was an error in the solution and that the students could find it themselves. At other times the teacher was to point out specific errors, including false assumptions, mistaken computations, and the mis-use of mathematical notation.

11. When a group finished solving a problem, the teacher was to check the solution unless the group members preferred to erase their work and go on to the next problem. However, for certain very important problems, the teacher was to urge the group members to have their solution checked before erasing it.

12. The teacher was to have a supply of extra

problems ready for groups that moved at a rapid pace and finished the day's work early.

13. The teacher was to feel free to develop a friendly, social relationship with the students by discussing non-business matters upon occasion during the class period. This was, of course, not to be done in such a way as to impede the learning of mathematics.

14. The teacher was to behave in an equalitarian manner by dressing in an informal manner, by calling the students by their first names, and by encouraging the students to call him by his first name if they could feel comfortable in doing so. He would not try to be "just one of the boys," but he would also not be particularly concerned over maintaining his professional dignity and status.

3.4 COOPERATION OR COMPETITION?

After specifying the leadership style of the teacher, it seemed appropriate to focus attention upon the functioning of the work groups in the small group-discovery class. The first issue considered was that of cooperation or competition within each work group. In a particular work group, students might compete with one another for various reasons--to be the first one to understand a problem, to show how smart they were, to gain the approval of the teacher, or to receive a higher grade for participation in class, if such were the policy. On the other hand, there could be a cooperative atmosphere in which the students work together in order to achieve a joint solution to the mathematics problems. In making the decision about cooperation or competition, the author was aided by previous investigations in the field of group dynamics.

A Pertinent Investigation

A theory of cooperation and competition has been developed by Morton Deutsch (1960). In the competitive situation, according to Deutsch, individuals work at cross-purposes; the concern of each individual is to advance his own status above that of others. Thus, when one individual progresses toward his goal, the progress of the others toward their goals is reduced. In the cooperative situation, on the other hand, the concern is for the accomplishment of the mutually-held goal. When one individual progresses toward the common goal in a cooperative situation, the other members are "carried along" toward the same goal.

Deutsch conducted an empirical study of the effects of cooperation and of competition in a classroom setting.

In the study, small discussion groups of students at MIT were to obtain group solutions of puzzle problems and of human relations problems. In the cooperative situation all members of a particular group received the same grade for their efforts. In the competitive groups, members were graded individually on a rank-order scale from best to worst. The following conclusions were obtained.

1. Organization. In the cooperative situation, as compared with the competitive situation, there was greater coordination of effort. Less time was wasted through needless duplication of the actions or ideas of others.

2. Motivation. In the cooperative situation, as compared with the competitive situation, members were more strongly motivated to accomplish the common goal.

3. Communication. Communication was better in the cooperative situation than in the competitive situation. In the cooperative situation members were more attentive in listening to the ideas of others. There were more understanding and acceptance of the ideas of others and more agreement with those ideas.

4. Orientation. In the cooperative situation, as compared with the competitive situation, individuals knew more about the attitudes of the other active group members.

There was greater group orientation in the sense of common awareness of "where they are and where they are going."

5. Group Productivity. In the cooperative situation, as compared with the competitive situation, the productivity of the group discussion was higher in both quantity and quality. More ideas were agreed upon for solving the problem, and these ideas showed relatively greater insight and understanding of the problem.

6. Interpersonal Relations. Individuals were more friendly in the cooperative than in the competitive situation. Individuals felt that their ideas would be looked upon in a more favorable manner, and there was more encouragement of others and voicing of approval. Members felt a greater obligation to others and more desire for the respect of others. In addition, there was a higher evaluation by the students of the results of the group activity.

Some of the results of the Deutsch study can be restated with emphasis upon the competitive groups.

7. Motivation. In the competitive situation, as compared with the cooperative situation, members were more strongly oriented toward outdoing one another.

8. Organization. In the competitive situation there

was greater duplication of effort than in the cooperative situation.

9. Conflict. In the competitive situation, as compared with the cooperative situation, members were less helpful toward one another. There were more instances of obstructive behavior, of self-defensive behaviors, and of internal group conflict.

10. Communication. Communication difficulties were more prevalent in the competitive situation than in the cooperative situation. Members talked more but listened to each other less attentively. There was less understanding and acceptance of the ideas of others, and members felt that their ideas were frequently ignored. Agreement was more difficult to achieve.

11. Group Productivity. In the competitive situation, the productivity of the group discussion was reduced by the " . . . lack of coordination, communication difficulties, persisting internal conflict, and lack of group orientation" (Deutsch 1960, p. 423). Individual productivity, as measured by the number and quality of ideas contributed by individuals, was as high in the competitive situation as in the cooperative situation. However, in the competitive situation it was more difficult to combine the

ideas of individuals into a problem solution agreed upon by the entire group.

The results obtained by Deutsch have been confirmed in other investigations, according to Bany and Johnson (1964) and Hare (1962). The studies have shown that problem solving by small groups is better accomplished with group cooperation than with competition.

Decisions for the Pilot Study

On the basis of the previously cited studies, it was decided that a cooperative atmosphere would be desirable within each small work group of the calculus class. The teacher, as supervisor of the work groups, was to promote cooperation by making statements of the following sort as needed.

1. "The group goal should be not only to solve the problem, but to solve the problem in such a way that everybody understands the solution."
2. "You can feel free to ask questions at any time. If somebody understands the problem, he'll explain it as clearly as he can."
3. "Some of these problems are very hard, and you'll really have to work together to solve them."

4. "The point is to have a solution agreed upon by everyone, not four individual solutions."

5. "There is no need to blame anybody for making mistakes; this is not a contest and you are not scoring points."

6. "How about listening to each other a little more closely? Are you really disagreeing, or are you just saying the same thing in different words?"

7. "Try to build on each other's ideas. If you give part of the solution and somebody else contributes another part, then pretty soon you'll have it all solved."

8. "Sometimes you can use part of an idea. You don't have to reject the entire idea because part of it is wrong."

9. "Let's agree upon one approach for attacking the problem. Then, whether it works or not, you can still try out those other ideas afterwards."

10. "We're all in this together; if one person has a good idea, it helps the whole group."

It was further decided, in order to avoid interpersonal competition, that the teacher would not praise or criticize individuals for their work. He would look at the group solution without asking who was responsible for it and he would not give individual grades for classwork. Although cooperation could be fostered by giving group grades, it was decided in accordance with the Deweyan philosophy to avoid the use of grades unless all other means failed to promote cooperation. It appeared that the supervisory practices described above would furnish a sufficient impetus toward cooperation and away from competition.

3.5 CONFORMITY

According to the previous decision, there was to be a cooperative atmosphere within each work group of the small group-discovery class. The prospect of cooperation in problem solving, in contrast with the individualism presumably displayed in a competitive atmosphere, might arouse doubts about conformity pressure in the work groups. In fact, one could even regard conformity as an extreme form of cooperation--a caricature in which the group members behave like the infamous "organization man." The organization man, one recalls, was so "cooperative" as to avoid all disagreement or conflict, to hold no opinions of his own, to constantly seek out the opinions of others, and to thoughtlessly proclaim his agreement with those opinions, irrespective of their content. There have, of course, been less extreme examples of conformity in groups. In any case, it did seem appropriate to examine some experimental evidence about conformity pressure in groups, to assess the pertinence of that evidence for the small group-discovery class, to determine whether conformity pressure in the work groups was actually a threat, and to find ways to reduce conformity pressure if it appeared necessary to do so.

Studies of Conformity

Asch (1960) conducted an experiment to determine the effects of group pressure upon the distortion of individual judgments. In the experiment all the persons in a room were required to publicly state their judgments concerning the lengths of certain lines. In each room there was a majority of planted subjects who were instructed in advance to give prearranged wrong answers. The "naive" subject in each group then found the evidence of his senses contradicted by the opinions of a majority of the group members.

The naive subjects in the experiment either maintained their independent judgment or else yielded to group pressure and modified their announced judgments. The subjects who maintained their independence did so in three different ways. (1) Some subjects were independent and confident in their own perception; they vigorously resisted the opposition of the others. (2) Some subjects were independent but withdrawn; they acted in order to maintain their individuality as a matter of principle. (3) Some subjects were independent but exhibited a large amount of tension and doubt; they preserved their judgment because they felt it necessary to deal with the task.

The subjects who yielded did so in three distinct ways. (1) A very few subjects yielded completely by distorting their perception to agree with the majority. They believed that the majority was right and were unaware of their own distortions. (2) Most of the subjects distorted their judgment by deciding that the majority was right and that their own perception was not accurate. These people felt doubt and little confidence in themselves. (3) Some subjects accepted their own perception as correct but still agreed outwardly with the majority view. It was important to these people to not appear different from or less competent than the others.

Asch found that if one other person agreed with the dissenter, the distortion of judgment was considerably reduced and sometimes eliminated. A single individual faced with a unanimous majority of three yielded as frequently as an individual faced with a majority larger than three. A majority of two produced less frequent yielding.

Cartwright and Zander (1960) have reviewed studies in which the strength of uniformity pressures was increased in order to further the achievement of the goals of the group. Uniformity pressures were increased when the group goal was clear, highly valued, or seemingly attainable, or when similarity of behavior seemed to be necessary in order to achieve the goal.

Deutsch and Gerard (1960) found that conformity to the judgment of other group members was decreased when the individual did not feel that others were putting pressure upon him to conform. Moreover, conformity to the judgment of others was reduced if the group developed standards which encouraged the individual to conform to his own judgment.

Decisions for the Pilot Study

Consideration of the study by Asch indicated that conformity pressure in the work groups of the calculus class was a danger to be reckoned with. However, it was recognized that not all conformity should be avoided, for conformity with certain standards of behavior seemed to be both desirable and necessary. For example, group progress would be hindered if the members consistently came late to class or refrained from doing the necessary homework.

The author was mainly concerned about student conformity while solving mathematical problems. There was the danger that group members might unconsciously distort, consciously modify, or simply not state their own opinions, in order to avoid disagreement with the views of others. If erroneous attempts at solutions were left unchallenged and if correct ideas were withheld, then group productivity would be reduced. If certain members let a problem solution "go by" and pretended to understand it, then individual learning would be reduced. Hence, it appeared that conformity in problem solving, in addition to making some of the group members tense and uncomfortable, could have detrimental effects upon both group productivity and individual learning. Finally, the promotion of independent thinking by students was one of the major goals of the small group-discovery method, but independent thinking and conformity were antithetical to one another.

A decision was made in accordance with the study by Deutsch and Gerard (1960) to reduce student conformity in problem solving by developing group standards which encouraged members to conform to their own judgment. This could be done through discussions with the entire class and through supervision of the work groups.

1. In discussions with the entire class or with particular groups as needed, the teacher was to emphasize the importance of independent judgment, the legitimacy of disagreement, and the obligation of group members to give reasons to support their statements.

2. When observing the small work groups, the teacher was to look for expressions of puzzlement or confusion, particularly on the part of members who were not giving opinions or requesting information.

3. If several group members were putting undue pressure on a single dissenting member, the teacher was to act as a mediator. He could solicit ideas from hitherto silent members, request that complete explanations be provided by both sides, give hints or ask certain leading questions, perhaps confirm the position of one of the two sides, or try to lower the level of tension in the group.

4. The teacher was to emphasize the distinction between thoughtless conformity to the views of others and a change of opinion based upon a thoroughly understood argument. He would attempt to discourage conformity in

problem solving without promoting the other extreme of stubborn refusal to listen to the views of others.

There was one further decision about conformity in the work groups. On the basis of the previously cited article by Cartwright and Zander (1960), the author inferred that competition between groups would increase conformity pressure within each group. It appeared that competition between groups would impel the students to work as fast as possible in order to outdo the other groups. The clear goal of winning the contest--at least in those groups which had a chance of winning--would call for certain uniformities in behavior. It seemed that the members would be under heavy pressure not to ask many questions, not to give detailed explanations, not to propose alternative solutions to the problems, and not to argue with the primary problem solver unless they were quite certain of their views. Thus, the pressure to avoid impeding the rapid progress of the group would oppose the attainment of maximal understanding by all the group members. Accordingly, it was decided not to foster competition between groups.

In summary, student conformity in problem solving was to be reduced through the development of standards which encouraged members to conform to their own judgment. In discussions with the entire class and with particular work groups, the teacher was to emphasize the legitimacy of disagreement and the importance of independent judgment and of reasoned argument. The teacher was to intervene as a mediator when there were indications of conformity pressure in particular groups. Competition between groups was not to be encouraged.

3.6 STUDENT LEADERSHIP IN THE WORK GROUPS

In addition to the variables of cooperation and conformity pressure in the small group-discovery class, there was the variable of student leadership. Should each work group have a designated student as leader? Before attempting to answer this question, it was necessary to inquire about the meaning of the term "leadership" in modern social psychology.

Leadership through the Performance of Group Functions

According to Cartwright and Zander (1960), modern studies in group leadership have tended to place emphasis upon various group functions to be performed by one or more of the group members. Group functions have been classified into two basic categories, termed "task functions" and "maintenance functions." The performance of task functions enables the group to make headway toward the achievement of its goal. Examples of task functions are the following: contributes ideas for the problem solution, suggests a plan of procedure, identifies information pertinent to the task, keeps the attention of members on the goal, integrates several ideas into a solution. On the other hand, functions involving interpersonal relations and the maintenance of the group as a social unit are termed "maintenance functions." Examples of maintenance functions are the following: provides encouragement, agrees with others, exhibits satisfaction with the work, solicits the views of a group minority, attempts to resolve conflicts, shows tension release. A person is said to exercise leadership when he performs group functions of either type.

According to Cartwright and Zander (1960), group functions may be distributed among some or all of the group members, or there may be a concentration of functions in a single person. However, Bales (1965) and others have shown that in discussion groups persons who perform a large number of task functions do not usually tend to also perform maintenance functions. In fact, the aggressive performance of task functions may provoke disagreement and hostility, in which case the maintenance functions tend to be performed by one or more persons not so heavily involved in the task. In order to maintain the important distinction between task functions and maintenance functions, the term "task leader," rather than simply "leader," is used in the discussion which follows.

The Issue for the Small Group-Discovery Class

The discussion now returns to the original question. It would be possible to appoint or to elect a task leader for each work group in the calculus class and to thereby create a status hierarchy. The work group would have one member with high status (the task leader) and other members with low status (the followers). The task leader could be designated for an entire semester or until the next change of group membership, at which time a new task leader could be appointed or elected. Thus, there were possibilities for either mobility or non-mobility in the status hierarchy. The opportunity for mobility could be increased if each small group had the option of deposing its leader and replacing him with another.

In deciding whether or not to have designated task leaders for the work groups, the author preferred to see some conclusive experimental evidence. Such evidence might show that in small problem-solving groups of college students, the use of a designated student as task leader had certain specific effects upon group productivity or morale. Unfortunately, no such evidence was available, and the decision was made on the basis of the following somewhat pertinent articles by Kelley (1960) and by Cartwright and Zander (1960).

Studies Involving Hierarchies of Power or Status

Kelley (1960) studied the effects of status and mobility upon communication in task groups. In a very brief experiment, Kelley created a two-level status hierarchy in groups of college students, who were to perform a complex bricklaying task and to communicate only with written messages. At the very beginning of the experiment, the members were assigned to positions having (1) either high or low status and (2) mobility or non-mobility. Subjects assigned to a state of non-mobility were told that they would retain their present high or low status throughout the experiment. Subjects assigned to a state of mobility were told that they might be shifted from low status to high status or vice-versa. The following results were obtained.

1. The "unpleasant" positions in the hierarchy were those of low status without upward mobility and high status with the possibility of demotion. Occupants of unpleasant positions tended to communicate about matters unrelated to the task; this apparently served as an escape mechanism.

2. In communication with persons of low status, persons of high status tended to restrict expressions of confusion with the task or criticism of their own job. Apparently, the high-status persons wished to maintain an image of competence in their position.

3. Persons in one level of the hierarchy felt restraints against criticizing persons at the other level. However, high-status persons felt more free to express criticism directly to low-status persons than vice-versa.

4. Occupants of the unpleasant positions experienced a decrease in the attractiveness of the group. The explanation was that high-status persons facing possible demotion were hostile toward those who threatened their position, and that low-status persons were hostile toward those in a "coveted but unattainable position."

5. Occupants of the unpleasant positions were markedly less friendly and offered less encouragement and praise than did individuals in the other two positions and in the control group, which had no status hierarchy.

Cartwright and Zander (1960) have cited various studies in which there were formalized differences in power or status among the group members. In some of the studies it was found that individuals having high power or status in a group felt more free to participate in discussions than did individuals with low power or status. Moreover, within hierarchies of power or status there was often an "upward orientation" in which subordinates directed communication to their superiors, seeked to associate with their superiors, and needed to feel liked by their superiors. Finally, efficiency in group problem solving was reduced in some instances when the group failed to make use of the correct ideas of members with low status in the hierarchy.

Decisions for the Pilot Study

For the sake of argument, it was assumed that each work group in the calculus class had a specified task leader and several followers. From the article by Cartwright and Zander (1960), the author inferred that the task leader would assume primary responsibility for solving the problem, that he would talk more than the other members, and that the other members would talk more to him than to each other. Hence, contrary to the Deweyan philosophy, the opportunity for learning by active participation would be reduced for the followers. It also appeared that the potentially productive contributions of some of the followers would too frequently remain either unspoken or neglected by the group, and that the lack of these contributions might reduce efficiency in problem solving.

Still assuming the existence of a designated task leader in each group, the author drew the following plausible inferences from the very brief study by Kelley (1960). Those followers who were frustrated leaders might tend to spend too much time discussing task-irrelevant material; this would signify a low level of personal involvement with the task. Moreover, the task leader might be overly concerned about maintaining his image of competence, and in order to protect that image, he might be unwilling to admit to his mistakes or to his confusion about certain problems. In addition, the mistakes of the task leader might remain too long uncorrected, due to the reluctance of the followers to criticize a person with higher status. If mobility were not possible, the above effects might persist over time, and the communications pattern might begin to resemble that of a lecture presented by the task leader to the followers. If mobility were possible, there might be hostility between the task leader and those who wished to depose him. Finally, the would-be leaders might strive to impress other group members and the teacher so that they could become task leaders in the future; hence, competition could be increased, contrary to a previous decision.

Although the experimental evidence was not conclusive, the designation of a task leader for each work group appeared to be a practice involving considerable risk. Moreover, there was no clear-cut case to establish the need for an appointed or elected task leader in each group. Accordingly, it was decided that the work groups were to operate without designated task leaders.

Studies of Power Relations and Emergent Leadership

The decision to operate the work groups without designated task leaders did not eliminate the possibility that certain group members would be more influential than others. Indeed, one of the familiar findings in group dynamics has been the following: in a group having no formalized hierarchy of status or power, there may still be large differences in the power of individual members to influence one another. These differences have been observed in a variety of settings, and some of the more striking confirmations have occurred in children's camps (Lippitt et al. 1960). In the camps it was found that certain group members had a great influence upon the opinions and behavior of others. Average group members tended to initiate "deferential, approval-seeking behavior" toward the more powerful group members. Moreover, average group members tended to spontaneously imitate the behavior of more powerful members; this phenomenon was termed "behavior contagion."

Carter et al. (1960) and Bales (1965) have observed the occurrence of a struggle for leadership in groups having no designated task leader. In some but not all of the groups, a particular person became so influential as to emerge as the clear leader with respect to the task. The individual who emerged as task leader acted in a forcible and energetic manner to support his own ideas against those of others. Moreover, Bales (1965) and others found that when hostility was generated among the contenders for task leadership, there emerged a "social specialist" who was concerned about interpersonal relations and the maintenance of the group as a social unit. Bales further found that certain task men talked too much and retained too much control over the discussion; these people were actively disliked. In addition, there were other individuals who talked too much but were not very competent on the group task; these people were also actively disliked by the other group members.

A Further Decision for the Pilot Study

The previously cited studies on power relations and emergent leadership dispelled any illusions about the possibility of a completely equalitarian work group in which all members had exactly the same influence. This condition appeared to be quite unattainable in practice. Nevertheless, it did seem possible to place some limitations upon the discrepancy in power between the least active and the most active group members. It was expected that the emphasis upon group cooperation would to some extent alleviate the struggle for task leadership. In addition, no person was to be allowed to dominate the discussion in a manner that would exclude or severely limit the contributions of others.

If necessary, the teacher was to influence the dynamics of particular work groups by drawing certain members into the discussion, by cautioning those who tended to dominate the discussion, by suggesting that different people assume primary responsibility for writing the solutions to different problems, and by stressing cooperation through means which have already been described.

3.7 GROUP SIZE

It has so far been decided that the work groups in the small group-discovery class were to function without definite leaders and that the atmosphere in each group was to be cooperative rather than competitive. Another variable which could influence the interaction of the group members was the size of the group itself. It seemed possible that the size of the group might have an effect upon the level of tension of the members, the amount of disagreement and antagonism, the level of cooperation, the inclusion or exclusion of members from the discussion, and the need for a definite leader. These possible effects have been empirically studied, and the results were used by the author in making decisions about group size.

Confirmation of an Earlier Decision

Discovery learning in mathematics could take place with the teacher leading a discussion of the entire class, rather than with the use of small work groups. As a matter of fact, both Cummins (1960) and Larsen (1961) have successfully taught calculus via a teacher-directed discussion of the entire class. However, the author believed that students would participate more actively and be generally more satisfied with the results obtained in small groups as opposed to the discussion of the total class. Some empirical evidence was available on this point.

Hare (1952) conducted an experiment on group size with discussion groups of boys. In the two experimental variations of the group size, the groups had five and twelve members, respectively. The following deficiencies appeared in the group with twelve members as compared with the five-member group. In the larger group, less consensus was attained in the discussion and there was less change of opinion. Members participated less frequently, felt that their own opinions were not important, and were less satisfied with the discussion. Members believed that there was not enough time for an adequate discussion and not enough opportunity for individuals to speak. There was a greater tendency for the large group to fragment into factions, and increased skill in leadership was needed to obtain agreement.

The study by Hare (1952) provided some support for the use of small groups. The case could also be made by means of the following argument, which assumed that in group problem solving only one person would talk or write on the blackboard at any given time. In Chapter II, emphasis was placed upon learning as both a social process and an active process. In the 2-member group--the smallest social group--each member could participate overtly by speaking and/or writing for 50 per cent of the time, provided of course that each of the members participated to the same extent. If all members of a 12-person group participated to the same extent, there could be only 8.3 per cent of the "air time" available for each person. Clearly then, the opportunity for active participation would decrease as the group size increased.* It thus seemed desirable to maintain work groups of a small size, and consideration was given to work groups having 2, 3, 4, 5, 6, and 7 members, respectively.

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*This argument was not intended to prove that a person could not learn by listening or by thinking silently.

The Two-Person Group: A Study and a Decision

Certain special characteristics of two-person discussion groups (dyads) have been empirically determined by Boles and Borgatta (1961). The dyads were marked by high rates of tension but low rates of disagreement and antagonism. The members frequently asked each other for opinions but tended to refrain from giving their own opinions. In the dyads the formation of a majority demanded unanimity; the problem of deadlock was a severe one because either member could bring the group to a halt by disagreement or withdrawal. There was no third party, mediator, or arbitrator in the resolution of differences. Despite the high tension level, hostility tended to be controlled, since the members had to continue to face each other.

The author speculated about the possible results of using two-person work groups in the small group-discovery class. On the one hand, the students were expected to solve problems by generating and evaluating many ideas. On the other hand, the dyads were known to be marked by a tense, cautious atmosphere in which the members avoided conflict and avoided expressing their ideas. It therefore appeared that the actions needed for effective problem solving could not very readily take place within the setting of a dyad. Moreover, a class of reasonable size would contain many dyads, and the teacher could spend only a small amount of time with each dyad to give needed hints and to mediate differences of opinion. Hence, periods of non-productivity, including deadlocks, might persist for too long a time in the dyad. In addition, if one member of the dyad were absent, the other member would be left without a work partner. Finally, the author believed that the presence of at least one more group member would lead to improved error detection. Although some of the opinions just expressed could better be regarded as researchable questions than as settled issues, there seemed to be sufficient justification for not using dyads in the small group-discovery class.

Further Studies of Group Size

Investigations by Mills (1960) and by Bales and Borgatta (1961) have shown the tendency of a three-person group to break up into a pair and an isolated member. Any single member of the threesome was threatened by solidarity between the other two members, and the strength of the two-person coalition was increased if the members united in opposition to the third party. On the other hand, the third party sometimes disrupted the solidarity of the other two, particularly when a conflict developed with the coalition. In the conflict situation, the third party might hold the balance of power or serve as mediator. In certain cases there were frequent splitting and formation of new coalitions, but in other instances a single coalition persisted for a long period of time.

Bales and Borgatta (1961) found certain differences between groups with an even number of members (4,6) and groups with an odd number of members (3,5,7) in the discussion of human relations problems. In the groups of even size, in comparison with the groups of odd size, the members tended to display more disagreement and antagonism and to ask for suggestions less frequently. The explanation was that groups of even size could split into two subgroups of equal size and thereby produce a protracted argument or deadlock, whereas the groups of odd size tended to split into a majority and a minority. It should be noted that in the groups of even size (4,6) there was not the same pressure to avoid conflict that occurred in the dyad, because a strict deadlock of 2 against 2 or 3 against 3 still left each member with a supporter.

An experiment in group size was conducted with college students in groups having respectively 2, 3, 4, 5, 6, and 7 members (Slater 1958). There were several groups of each size discussing human relations problems. The members of the groups were asked whether the size of their group was too large or too small for effective problem-solving. No member of a 5-person group reported that his group was either too large or too small. In the groups with 2, 3, or 4 members, there were significantly many reports that the group was too small. In groups of 6 or 7 members, there were significantly many reports that the group was too large.

Disadvantages of the larger groups (5,6,7) were specified by Slater. The members were "too aggressive, impulsive, competitive, inconsiderate," and the group "too hierarchical, centralized, and disorganized." The members reported that the group worked inefficiently, that certain people were excluded from the discussion, and that a definite leader was needed to keep things moving. On the other hand, in the smaller groups (2,3,4) the members were "too tense, passive, tactful, constrained." There was too much fear of mutual antagonism. However, these effects decreased with increasing size of the smaller groups.

Decisions for the Pilot Study

In making the decision about the size of the mathematics work groups, it was necessary to recall that these groups were intended to function without having definite leaders. The group atmosphere was intended to be cooperative rather than competitive, and no members were to be excluded from the discussion. The Slater experiment showed that these conditions were not likely to be achieved in the larger groups, particularly in those having 6 or 7 members. The group of 5 members was included as a possibility for the mathematics work group, since the 5-member group was not reported to be either too large or too small in the discussion of human relations problems.

The experiments on group size indicated that the mathematics work groups should not have 2, 6, or 7 members. It was not clear, however, whether the optimal number of members was 3, 4, or 5. The advantages and disadvantages of the 3 group sizes were as follows:

a. The 3-member group (triad) would permit the largest amount of individual participation among the groups under consideration. However, the triad was marked by a tendency toward the formation of coalitions which excluded

1 member. Moreover, the absence of any member of the triad reduced the group to the uncomfortable size of 2 members.

b. The 4-member group would allow the intermediate amount of individual participation among the groups under consideration. The 4-member group could split into 2 subgroups of equal size and thereby produce a prolonged argument or deadlock.

c. The 5-member group would allow the least amount of individual participation among the groups under consideration. The 5-person group entailed the possible dangers of competition, exclusion of members from the discussion, and the need for a definite leader. However, the 5-member group was the reported preference in the discussion of human relations problems.

The experimental evidence failed to indicate the clear superiority of any of the 3 choices, so other factors were taken into account in making the decision. The various studies on group size involved the discussion of human relations problems, rather than mathematical problems. The objectively demonstrable solutions of the mathematics problems, as compared with the value-laden human relations tasks, might alter the interaction in the work groups. Certainly, the reported preference for 5-person groups when solving human relations problems could not be assumed to apply in groups solving mathematical problems. Moreover, in the mathematics work groups but not in the human relations discussion groups, the students were to write their problem solution on the blackboard. A 5-person group fitted into a relatively small space about one portion of the blackboard could be crowded enough to "crowd someone out." Bearing in mind also the restricted opportunity for participation in the 5-person group as compared with smaller groups, the author decided not to make use of the 5-person group in the pilot study.

In the various studies on group size, no outside influence was present in the leaderless experimental groups. However, in the mathematics work groups, the teacher was to appear repeatedly during the class period. In a triad of students, the presence and influence of the teacher might either inhibit or shift the formation of coalitions. In a 4-person group, the presence of the teacher would help resolve deadlocks and might reduce the tendency to split into 2 subgroups of equal size. However, these effects could not be predicted with any certainty.

There seemed to be no clear-cut case for selecting either the 3-member group or the 4-member group. Further research was clearly necessary to determine the optimal size of the mathematics work group. However, a choice had to be made, despite the lack of conclusive experimental evidence. The choice for the pilot study was the 4-member work group.

In summary, the experimental evidence indicated that a mathematics work group should not have 2, 6, or 7 members. An argument concerning the special nature of the mathematics work group was used to rule out the 5-member group. There was no clear case for selecting the 4-member group instead of the 3-member group.

It remained to specify the number of work groups in the small group-discovery class. It was not clear that the teacher could give adequate attention to more than 3 work groups during the first trial of the new instructional method. Accordingly, the pilot study was to be conducted with a total class of 12 students, who worked in 3 small groups, each of which had 4 members.

3.8 GRADING POLICIES

The last variable for discussion in this chapter is that of grading policies for the small group-discovery class. It appeared that grades could have an effect upon the level of student anxiety, which in turn might have an effect upon student performance in solving some rather challenging mathematical problems. There was some experimental evidence available on the effects of grades and anxiety upon learning and problem-solving performance.

The Effects of Grades and Anxiety upon Learning and Problem Solving

Kilpatrick (1969) has cited the results of a study by Gangler on the effect of grades upon learning and problem-solving performance. " . . . College students who were informed that their work on a series of learning tasks in symbolic logic would count toward their mathematics course grade performed less well on learning and problem-solving tasks than students who were not so informed. The effect was greater for students of high intelligence than for students of low intelligence" (Kilpatrick 1969, p. 157).

According to Symonds, " . . . conditions within a classroom that induce mild anxiety will have the least negative effect or even a positive effect on rote learning, but these conditions will interfere increasingly with problem solving, reflective thinking, and creativeness. If we want to teach pupils to think, then it is essential that anxiety be eliminated from the learning situation" (Symonds 1958,

p. 14). This statement summarized the results of a number of studies with several kinds of learning tasks under varying conditions of stress or anxiety.

McKeachie (1954), after reviewing several studies of student-centered versus teacher-centered instruction at the college level, arrived at the following conclusion. "It seems significant that almost every one who tries student-centered teaching methods finds that the problem of grades presents the greatest obstacle to success" (McKeachie 1954, p. 149). Moreover, in two of the studies showing positive results with student-centered instruction, the teacher reduced his own power by turning over to the group the "responsibility for determining grades."

Confirmation of Some Previous Decisions

The studies just cited appeared to be quite pertinent for the design of the small group-discovery method, since this method was planned to be a student-centered approach with a heavy emphasis on problem solving. In order to avoid competition, it was previously decided not to give individual grades for work done in class. In accordance with the Deweyan philosophy and with the studies on the democratic style of leadership, it was previously decided not to give examinations in class, to use an A-B grading scale for the calculus class, and to leave further determination of grading policies up to the students as a group. The empirical results cited by Kilpatrick (1969), Symonds (1958), and McKeachie (1954) provided additional support for these earlier decisions.

3.9 A SUMMARY OF DECISIONS

On the basis of a number of experimental studies in group dynamics, various decisions were made concerning the social climate in the small group-discovery class. There follows a summary of the major decisions.

A. The teacher was to use the democratic style of leadership, not the autocratic or laissez-faire styles.

   1. Most policies in the calculus class were to be arrived at through group discussion and decision-making by a majority vote. The teacher was to serve as discussion moderator.

   2. A perspective on each day's mathematical activities was to be provided by the teacher at the beginning of each class period in a discussion with the entire class. These discussions would generally last no more than five or ten minutes, during which time the teacher was to present new concepts, raise questions, answer certain questions, assign problems, give hints if necessary, and serve as discussion moderator. There would sometimes be brief discussions at the end of the class period.

3. After an initial period of acquaintance, the students would be free to decide upon the membership and division of labor in the several work groups.

4. The teacher was to give only a minimal amount of objective, constructive praise and criticism, usually directed to the group as a whole, rather than to particular individuals.

5. The teacher was not to give orders or disrupting commands.

6. The teacher was to keep track of the progress of the work groups and to offer suggestions at times when they would be needed and presumably appreciated.

7. Guiding suggestions of a mathematical nature were to be given in the form of hints, sometimes using the heuristic techniques of Polya.

8. The teacher would sometimes offer guiding suggestions with respect to the organization of work within a particular group and with respect to the functioning of the group as a social unit.

9. The teacher was to provide technical information upon request, if the development or recall of that information was not judged to be an important aspect of the problem at hand.

10. The teacher was to stimulate self-direction by encouraging individual members to think through and elaborate upon their ideas and to detect their own errors. However, if necessary, the teacher was to point out specific errors, including those of logic, computation, or notation.

11. When a group finished solving an important problem, the teacher was to check the solution. For not-so-important problems, the teacher might or might not check the solution, depending on the group preference.

12. The teacher was to have a supply of extra problems ready for groups that moved at a rapid pace and finished the day's work early.

13. The teacher was to feel free to develop a friendly, social relationship with the students by discussing non-business matters upon occasion, but not to the detriment of learning.

14. The teacher was to behave in an equalitarian manner, including a reciprocal first-name basis with the students. The teacher would not try to be "just one of the boys," but he would also not be overly concerned about his professional dignity and status.

B. Within each work group, there was to be an atmosphere of cooperation rather than competition.

1. The teacher, as supervisor of the work groups, was to promote cooperation by making statements emphasizing matters such as the following:

   a. The goal of solving the problem in such a way that all members understand the solution

   b. The need for joint efforts in order to solve difficult problems

   c. The importance of listening carefully to the ideas of others

   d. The fact that one person's good idea helps the entire group

2. As previously stated, the teacher was usually not to praise or criticize individuals for their work in class.

3. The teacher was to check the group problem solution without asking who was responsible for it.

4. Individual grades would not be given for classwork.

C. Student conformity in problem solving was to be reduced through the development of standards which encouraged members to conform to their own judgment.

1. In discussions with the entire class and with particular groups as needed, the teacher was to emphasize

the legitimacy of disagreement, the importance of independent judgment, and the obligation of group members to give reasons to support their statements.

2. When observing the small groups, the teacher was to look for expressions of puzzlement or confusion, particularly on the part of members who were not giving opinions or requesting information.

3. If several group members were putting undue pressure on a single dissenter, the teacher was to act as a mediator in the discussion and to try to lower the tension level in the group.

4. The teacher would attempt to discourage conformity in problem solving without promoting the other extreme of stubborn refusal to listen to the views of others.

5. Competition between groups was not to be encouraged.

D. The work groups were to operate without designated task leaders.

1. There was no intent to create a completely equalitarian work group in which all members have exactly the same influence.

2. There was a definite intent to limit the discrepancy in power between the most active and the least active group members.

3. The struggle for task leadership was to be alleviated through emphasis upon group cooperation.

4. No person was to be allowed to dominate the discussion in a manner that would exclude or severely limit the contributions of others.

5. If necessary, the teacher was to influence the dynamics of particular work groups by the following means:

   a. Drawing certain members into the discussion

b. Cautioning those who tended to dominate the discussion

c. Suggesting that different people assume primary responsibility for writing the solutions to different problems

d. Using previously described techniques to promote cooperation

E. Each work group was to have 4 members, and the pilot study was to be conducted with a total class of 12 students, who worked in 3 small groups.

F. Grading policies were designed to minimize student anxiety.

1. There was to be an A-B grading scale.

2. There would not be any examinations in class.

3. Further decisions about the grading system were to be made by the students as a group.

CHAPTER IV

CLASSROOM ORGANIZATION AND SOCIAL CLIMATE DURING THE PILOT STUDY

4.1 OUTLINE OF CHAPTER IV

In the previous chapter a number of decisions were reported concerning the classroom organization and social climate in the small group-discovery method. Those decisions were intended as general guidelines to be followed during the pilot study. The purpose of the present chapter is to show how the guidelines were applied in practice. The chapter includes an anecdotal account of significant experiences pertaining to the classroom organization and social climate and to the dynamics of the learning groups. The major variables considered in the present chapter are (1) the selection of the students, (2) the leadership style of the teacher, (3) cooperation versus competition in the work groups, (4) leadership by students, (5) conformity pressure, and (6) group size. Mathematical examples are included in the present chapter only if they pertain to one or more of the cited variables.

4.2 SELECTION OF THE STUDENTS

For the first trial of the new instructional method, it seemed desirable to work with students who were capable of learning calculus and who had non-negative attitudes towards mathematics. It appeared that putting the method into operation for the first time would involve difficulties enough, without adding the handicap of students who were relatively poor at mathematics or who disliked the subject. Other requirements were added to guarantee that the students were somewhat similar in age and in mathematical background. In all, there were five entrance requirements for the small group-discovery class. In order to be admitted, the student needed (1) to be a freshman or a sophomore, (2) to have no or very little previous knowledge of calculus, (3) to have studied high school mathematics through trigonometry, (4) to have grades of A or B in his high school mathematics courses, and (5) to be at least mildly interested in mathematics.

At the University of Wisconsin, students are placed in multi-section courses through a system of assignment committees. The author of this paper, who was also the teacher of the class, sat at one of these committees with a sign reading "Small Calculus Section." Regular members of the assignment committee examined the records of incoming students and suggested to certain students with good mathematics records that they inquire about the small calculus section. Other students noticed the sign themselves and came up to inquire about the class.

Students were selected for the class on the basis of 10-15 minute interviews with the teacher at the assignment committee. In each interview the teacher explained that the class would function without either lectures or a textbook and would involve the solution of mathematical problems by students working in small groups. The student was told that some of the problems would be difficult, that a rather large amount of homework would be required, that participation in class was expected, and that he should join the class only if he enjoyed working with other people. It was further stated that the class was to continue for an entire academic year, that material would not be covered in the same order as in the regular sections, and that a student who began the class during the first semester would find it difficult to transfer into another class for the second semester.

Each prospective student had questions to ask the teacher. The student usually wanted to know if as much material would be covered in the special section as in the other sections, and he also inquired about the method of grading. The teacher provided assurance on the first point but did not discuss the method of grading, except to say that he had never been noted as a very hard grader.

Only one student who was interviewed decided not to join the class. Certain students made their decisions immediately, others hesitated for an hour or so, and one or two people thought about the matter overnight. The initial student reactions ranged from mild skepticism to cautious optimism to marked enthusiasm about the class.

The class began with the twelve students who volunteered at the assignment committee. However, within the first two days of class, two students dropped out--one because he felt he could not get along without a textbook, and the other because she thought the class would not cover enough material and would be slowed down by certain less able class members. The vacancies were filled within two days by friends of certain class members, who were introduced to the teacher and who went through the same interviewing process as the other members.

In summary, the students in the calculus class were all volunteers who were at least mildly interested in mathematics and who had grades of A or B in their high school mathematics courses. There were eleven freshmen and one sophomore; eight students were male and four were female. In addition, five of the twelve students took the course for honors credit. The twelve students continued in the class for the entire academic year 1967-1968, meeting for five days per week.

4.3 THE LEADERSHIP STYLE OF THE TEACHER

During the pilot study, the teacher of the calculus class made use of the democratic style of leadership. A description is now presented of the practical application of that leadership style. The major aspects of the leadership style are discussed in the same order as in Chapter III.

Group Discussions and Decision-Making

Most policies in the calculus class were determined through group discussion, and group decision-making by a majority vote. One of the first issues discussed was that of grades.* During the first week of class, the teacher mentioned his previous decision to use an A-B grading scale, not to give examinations in class, and to leave further determination of grading policies up to the group. He mentioned that this was done to minimize grading pressure and to allow the students to freely express their opinions, and he expressed confidence in the class members by noting that they all were good students of mathematics who would probably earn at least a B in any calculus class. He suggested that with these guidelines, further discussion of grades could be avoided for several weeks, until the students had a clear idea of what the class was all about. However, the students wanted the issue settled at once.

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*The democratic leadership style includes the practice of group discussion and decision making. Although particular decisions by the group are not aspects of the leadership style, it is convenient to discuss those decisions in conjunction with the leadership style.

A group discussion was held, and the class members proposed the following alternative grading schemes, which were written on the blackboard to clarify the alternatives. The choices were: (1) take-home exams, (2) several short quizzes in class every week, (3) homework problems frequently assigned, (4) a grade for classwork determined by the teacher, (5) a grade for classwork determined by the students, (6) group projects done outside of class, (7) problem sets solved in groups outside of class, (8) complete determination of grading policies by the teacher, (9) a combination of the previous choices,

(10) self-grading, (11) freedom for each person to choose his own grading scheme from among the previous choices.

There was a heated discussion of the various alternatives, during which the teacher refused to make the decision himself and also refused to assign students grades for work done in class, on the grounds that such a policy would foster competition. After a discussion engendering a very tense atmosphere and a substantial amount of conflict, the students decided to base the grade upon take-home exams and problem sets solved in groups outside of class. They further specified that any disagreement over grades was to be ironed out in a conference between the teacher and the student whose mark was in question.

A second discussion of grading policies occurred during the fifth week of the first semester. It was necessary at the University of Wisconsin to give a six-week grade to all freshmen. The students decided that each student was to determine his own six-week grade, rather than having a take-home exam for that purpose, and that the student was to have a personal conference with the teacher to explain the reasons for his particular choice of grade. In addition, the students altered their previous decision, specifying that the entire grade after the six-week marking period was to be determined by take-home exams. They specified that the exams should be given every two or three weeks, that students were to work independently of one another, and that students might refer to their dittoed class notes while working on the exams.

In the conference about the six-week grade, the students used a variety of criteria for grading themselves. One boy gave himself a grade of A because he had contributed several good ideas to his work group. Another boy, who had also contributed several good ideas, assigned himself a B grade on the grounds that certain other class members were smarter than he was. Other students graded themselves on the basis of the amount of homework done, their natural ability, or their contributions in class.

The decision to base the course grade on take-home exams--euphemistically called "review sheets"--remained in force for both semesters. Three review sheets were given during the first semester, and four during the second semester. In some instances there was a time gap of more than three weeks between review sheets, due to the length of certain major content units.

The discussions of grading policy, although intense while they lasted, took less than two class periods during the entire year. During these and other discussions of group policy, the teacher tried to act more as a moderator than as a participant. He usually expressed an opinion only on suggestions which were in direct conflict with the underlying philosophy of the course or with established principles of group dynamics.

In addition to decisions about grading policies, other decisions were made by a simple majority vote of the students. (1) Office hours of the teacher were arranged by a vote indicating times when most students could come. (2) The students decided to change the membership of the work groups every two to three weeks or at the end of major content units. At the end of each unit of content, the teacher asked the class members whether they wished to change groups. The answer was usually affirmative; however, the students were always careful to refrain from criticism of their present group. They made statements such as the following: "It's nice to change every so often because you learn to work with new people." "We have good groups now, but you can learn a lot from seeing how some other people would attack the problems." (3) The students as a group voted upon the dates for the first two review sheets. However, it was impossible to select a mutually acceptable date for the third review sheet. Faced with this difficulty, the group decided that each student could take the review sheet home at a time most convenient for him during a period of four or five days. They also reaffirmed an earlier decision that review sheets were to be done individually, rather than cooperatively. These policies remained in force for the rest of the year.

(4) Although many people continued to work on problems after the bell rang, the group decided to end class officially at the bell so that those people with other commitments could leave. (5) After a lengthy and intense discussion, the group decided not to formulate definitions which would conflict with the standard mathematical definitions. One girl, speaking in favor of non-standard definitions and terminology, said: "This is our mathematics. We are doing it ourselves and there's no reason why we should stick to anyone else's definitions." This argument was countered by other students as follows: "Mathematicians are smarter than us, and they must have had good reasons for making up their definitions." "What's going to happen to us next year in a regular class if we don't know the same terminology that everybody else uses?" The particular subject matter at issue is discussed in Chapter V. (6) During

the second week of classes, the students told the teacher that the homework problems were too time-consuming and too theoretical in nature and that there should be more short computational problems. The students also requested an overview of the mathematics content for the first major unit of the course, which dealt with sequences. The request was made, apparently, "to make sure we're getting somewhere, not just talking about any old topic that happens to be interesting." The students wanted to be sure of the existence of some plan for covering the material in the unit. In compliance with the student opinions, the teacher presented an outline of a unit plan and selected some short computational problems for homework. For the rest of the pilot study, some short computational problems were included in the homework assignments; however, after the first two units the students no longer seemed to want a unit plan.

In conclusion, a number of decisions regarding group policies were made by the students in the small group-discovery class. In addition to group decision-making, the application of the democratic leadership style entailed a number of other practices, which are described in the following discussion.

The Activity Perspective

A perspective on each day's mathematical activities was provided by the teacher at the beginning of each class period in a brief discussion with the entire class. During these discussions the teacher presented new concepts, raised questions, answered certain questions, assigned problems, gave hints if necessary, and served as discussion moderator.

The discussions with the entire class usually lasted for no more than five or ten minutes and were sometimes much shorter. Several examples of specific discussions are now presented.

a. Teacher: "Yesterday, after a great struggle, we developed a formula for the derivative of the composite function $h(x) = f(g(x))$. Who remembers how that formula turned out?"

Response: Various students suggested possible formulas, all of which were written on the board, without comment.

Teacher: "Here are four possibilities. Which one is it?"

Response: The students then identified the correct formula.

Teacher: "Okay, that's the formula written in functional notation. In our other notation using variables, we can let $y = f(u)$ and $u = g(x)$. The first problem for you in your groups is to restate the derivative formula using the notation with variables. Just one hint to get you started: if y = h(x), how do you write the derivative h'(x) in the notation with variables?"

Response: After a noticeable pause, someone said "h'(x) = dy/dx."

Teacher: "There's the notation you are going to use in rewriting the derivative formula. Today in your groups you'll also have a chance to apply that formula in practice. Here are some problems to try your hand on."

b. Teacher: "Today we'll have some more practical problems to show one of the uses of calculus. We already know how to find areas under curves, and today we're going to work with volumes. To set up the notation, let f(x) ≥ 0 for all x in [a,b]. If the curve y = f(x) is revolved about the x-axis, a surface is generated which looks like this (makes a drawing). Today's problem is to develop a method for finding the volume inside that surface. Who has an idea to get us started?"

Response: "We could put a little rectangle underneath the curve and see what happens when it goes around."

Teacher: "Let's see what your rectangle looks like. Can you describe its location a little more fully?"

Response: "The base is on the x-axis and the height runs up to the curve."

Teacher: "So there is one approach for finding the volume, and maybe that's enough of a start for today. The big problem for today is to develop the method for finding the volume inside the surface. While you work on that in your groups, I'll write equations of several specific functions on the blackboard. Once you have your general method, you can try it out in finding the volume inside those given surfaces. Just one more hint--draw some careful pictures."

c. Teacher: "In developing the formula for the derivative of a product, we came across the relationship $\lim_{x \to c} f(x) = f(c)$. That same relationship came up in developing several other derivative formulas. What was the name we used for that relationship?"

Response: After a long pause, someone said "continuity."

Teacher: "Could you specify it a bit more?"

Response: "The function was continuous at $x = c$."

Teacher: "That's it. By definition, f is continuous at c if and only if $\lim_{x \to c} f(x) = f(c)$. Now since the concept of of continuity came up several times in developing derivative formulas, perhaps there is some connection between continuity at a point and the existence of a derivative at that point. Can anyone guess what that connection might be?"

Response: "If f has a derivative at c, then f is continuous at c."

Response: "No, you've got it backwards. If f is continuous at c, then f has a derivative at c."

Teacher: "Well, there are two possible theorems; I'll write them both down. In deciding whether either or both are correct, let's try to think of some examples which violate one of the theorems."

Response: "We found that the absolute value function is continuous at x = 0 but has no derivative at x = 0."

Teacher: "So which theorem does that violate?"

Response: "The second one. It shows that a function might be continuous at a point without having a derivative at that point."

Teacher: "That rules out the second theorem. Can anybody think of an example which violates the first theorem?" (long pause)

Teacher: "Well, if nobody can think of a counterexample, then maybe it's true. The first problem for the groups today is to try to prove that theorem. Just one hint:

write out the hypothesis and the conclusion very carefully, and see if the same expression appears in both of them."

d. An occasional discussion with the entire class lasted as long as fifteen minutes; this usually happened only when there was new terminology to be introduced. For example, after the students had computed areas under certain curves as limits of sequences without using any terminology, it was necessary to introduce the notation for the Riemann upper and lower sums. It was also necessary to discuss the integrability of a function in terms of the agreement or non-agreement of the limiting values of the upper and lower sums. The students, incidentally, thought that all functions were integrable. Since the notation was rather complicated, the discussion took a full fifteen minutes.

e. During the entire pilot study, there was only one instance of a lengthy mathematical discussion with the entire class. The discussion lasted for twenty-five minutes, and the topic was the development of a method for computing the arc length of a curve given by parametric equations $x = f(t)$, $y = g(t)$. In the discussion the students quickly suggested that the length of the curve could be approximated by the lengths of polygonal line segments. When asked how to specify certain points on the curve in order to construct the n^{th} approximating polygon, the students made the following suggestions: (1) choose n line segments with equal length, (2) subdivide the x-axis into n subintervals of equal length, (3) subdivide the y-axis into subintervals of equal length. Not one student suggested the "desirable" approach of subdividing the t-axis into n subintervals, and the teacher had to make that suggestion himself. There was a lively discussion about the feasibility and possible equivalence of the varying approaches. After the students were convinced that the "peculiar" method of subdividing the t-axis might be a reasonable approach, they carried out the detailed derivation of the arc length formula in their work groups.

With a very few exceptions, almost all mathematical discussions with the entire class lasted for a maximum of ten minutes, and most of the discussions took less than ten minutes. The discussions were intended to set the stage for the main activity--student problem solving in the work groups. Enough input was provided in the discussions so that the work groups could function productively for the rest of the class period. It was interesting to note that almost everyone was actively involved in his work group, but only a few students participated in mathematical discussions of the entire class.

Student Choice of Group Members

The students determined the membership of their work groups, except during an initial period of acquaintance. During the first few weeks of the class, the teacher suggested that people change groups every two or three days and that they try to pick group members with whom they had not previously worked. Everyone wore a name tag during the first week of class. After the initial period of acquaintance, the students were free to decide upon the membership of the work groups and upon the time schedule for changing membership. As previously reported, the students chose to change groups every two or three weeks or at the end of a major unit of the mathematics content.

The process of changing groups was very brief, usually taking no more than three minutes. However, the process was very awkward, and for some people, rather painful. While some students appeared not to care whom they worked with, other students had certain preferences and anti-preferences. The students almost never verbalized a desire to avoid working with certain others; as a result of this lack of open-ness, people sometimes ended up working with other class members whom they disliked.

The students had various ways of coping with the threat of changing groups. One girl always went straight to "her corner" of the blackboard and waited for others to join her. Several people usually sat and pretended to do some homework until the groups were basically formed; they then looked around for a vacancy. One girl sometimes wandered around the room looking quite lost until she finally settled upon a group. On one occasion another girl moved very rapidly as far away as she could get from a particular boy; it turned out that the two had been dating but stopped seeing each other. In one instance a group formed within a very few seconds; this had been arranged by telephone the previous night. (There was a dittoed list of names, addresses, and telephone numbers of all the class members.)

The threat involved in changing groups was dramatically illustrated on one occasion when the students, after indicating that a change was desirable, sat without moving for a full two minutes. The teacher finally cajoled several individuals into making choices and the rest of the students followed along. After this incident, which occurred late in the first semester, the teacher asked the class members to write down names of group members they preferred working with and of other persons whom they preferred to avoid. He explained that their preferences would be kept confidential and that the intent was to make up a list of suggested working combinations from which the students could choose. However, most students turned in blank sheets of paper, and the attempt to make up a list of possible groups was a failure. One student later explained that despite the promise of confidentiality, certain non-preferences would be readily apparent if the two names never appeared in the same group. During the pilot study no means was found to facilitate the process of changing groups or to guarantee the formation of effective and satisfying working combinations.

Praise and Criticism

The teacher tried to use criticism and praise in an objective, constructive manner, and to criticize ideas when necessary, rather than people. There were, however, two instances involving personal criticism. In the first instance, the teacher said to a girl, " ----, you still don't know your derivative formulas." She immediately froze up and was noticeably less friendly to the teacher for the next week, during which time she also participated to less than her usual extent. In this instance, it would have been more effective and less hurtful to say to the group as a whole, "If you know those derivative formulas, you can solve problems quickly and more accurately." In the second glaring instance of personal criticism, the teacher said to another girl, "Why don't you look at your solution and point out your mistake." The sarcastic reply was, "Well now, if I'd known it was a mistake, I wouldn't have done it, would I."

In the two cited instances of departure from the democratic leadership style through the use of personal criticism, negative feedback from the students was immediate and strong. After these instances, the teacher refrained from criticizing people. Problem solutions were criticized in the sense that errors were pointed out, but no concern was shown as to who had made the error. Criticism and praise were generally directed to the work group as a whole, rather than to particular individuals. Praise was offered infrequently, usually only when a group came up with a problem solution previously unknown to the teacher. A number of examples of such solutions are presented in Chapter V.

Orders and Disrupting Commands

The teacher almost never gave orders. Occasionally, in the middle of a problem solution someone asked, "What do you want us to do now, Neil?" The reply was, "It doesn't matter what I want you to do; you just do what's necessary to solve the problem." If leadership was lacking in a particular work group at some time, the teacher got them started by asking a question, rather than by giving orders. For example, the order, "Compute that integral," was replaced by the question, "What techniques do you know for computing that integral?"

It was usually easy to avoid the use of disrupting commands. There was no need, for example, to tell a group to erase an attempted solution and start over. Instead, the teacher simply pointed out an error in the attempted solution. If the error was a fundamental one, the students quickly decided to try another approach, without any command from the teacher to do so.

There was one instance of a disrupting command in which the students were stopped in the middle of a proof of the chain rule for derivative functions. The proof was difficult, the students were apparently making no progress, and the teacher told them to stop and think about it that night. Although some students felt relieved, others expressed resentment during a period of several minutes with comments such as "I sure hate to get cut off in the middle of a problem." An incident of this type occurred only once.

Sometimes near the end of a class period, two of the three groups were finished with a problem set, but the third group was not. The teacher requested the members of the third group to stop working so that a summary could be given. This was a disrupting command since the members knew they were expected to comply. It became quickly apparent that students did not care to have a summary at the end of the period. The class was difficult to control, since the students kept talking to each other about the problem solutions, rather than attending to the summary. Several students stated that no summary was necessary, since they knew what they had done and required no further reiteration to understand it. Accordingly, the practice of having end-of-the-period summaries was abandoned, and activity in the work groups continued until, and often past, the sounding of the bell.

Guidance by the Teacher

The mathematical work was done by the students in their small groups, with help available from the teacher as needed. The students wrote their problem solutions on the blackboard, so it was easy for the teacher to keep track of the group progress. The teacher frequently did not wait for a request for assistance, but offered suggestions at times when they would be needed and presumably appreciated. In general the groups progressed independently of the teacher on short computational problems having definite answers. However, on theoretical problems involving the proofs of theorems the students needed a good deal of assistance. When the theorems were difficult, the teacher was barely able with considerable effort to provide enough assistance to the three groups. At other times it was no strain to provide adequate supervision.

Under most circumstances, a visit with a particular work group took less than one minute. Sometimes, a visit lasted only for ten or fifteen seconds--for example if it was only necessary to point out an arithmetic mistake, ask the reason for a particular step, check the solution to a rather simple problem, or inquire which of several possible techniques were being used to solve the problem. Upon occasion a visit with a particular group lasted for two or three minutes--when there was a very subtle point for discussion in a proof, when the approach to the problem was not clear to the students, or occasionally when the students could not understand the hints and required additional help. However, if a visit dragged on too long, members of other groups began calling the teacher for help. Finally, a very few class members always wanted to talk with the teacher about their ideas, rather than conversing with the other group members. When a visit was extended solely to talk about the ideas of one group member, the teacher usually made a remark such as, "That's an interesting idea; how about talking it over with your colleagues?"

Guiding Suggestions of a Mathematical Nature

Guiding suggestions of a mathematical nature were given in the form of hints, sometimes using the heuristic techniques of Polya (1965). There follows a number of examples of hints given to the work groups.

a. The teacher frequently used the heuristic technique of asking the students to concentrate upon the given data, the desired result, and relationships between the two.

In many proofs pertaining to the definition of the limit of a sequence or a function, the teacher asked, "What are you given and what are you supposed to prove?" He then suggested that the students write down the given hypothesis and the desired conclusion in close physical proximity to one another. This technique was effective in proving results such as the following: If $\lim_{n\to\infty} x_n = L$, then $\lim_{n\to\infty} (-x_n) = -L$. The students were usually reluctant to carefully record the given data and the desired outcome, and they were sometimes surprised to find a "difficult" problem practically solved just by recording and examining some pertinent information.

b. The teacher sometimes suggested the use of an analogy with previous results. For example, the students had difficulty in deciding whether the function $f(x)$ had no limit or two limits as x approached 0, where $f(x) = 2$ if $x \geq 0$ and $f(x) = -2$ if $x < 0$. The teacher suggested a comparison with the sequence 2, -2, 2, -2, 2, -2, ..., where the same issue had previously been settled.

c. It was sometimes helpful to suggest that the students consider a simpler instance of a general problem. When the students could not guess the formula for the derivative of a product of n functions, the suggestion was made to consider the simpler cases $n = 2$, $n = 3$, and $n = 4$.

d. It was sometimes possible to formulate a general result from consideration of several special cases. For example, the students correctly conjectured the fundamental theorem of calculus after having computed the three integrals $\int_a^b x^k\,dx$ for $k = 1, 2, 3$. (This approach was previously used by Cummins (1960).)

e. It was sometimes useful to suggest that the students discover or confirm results by drawing pictures. For example, the students were initially not certain whether the definite integral of a sum was the same as the sum of the definite integrals; the hint was given to draw graphs of two positive functions and to interpret the integral as an area. Similarly, when the students could not remember if $\frac{d}{dx} \cos x = \sin x$ or $-\sin x$, the hint was given to draw graphs of the sine and cosine functions and to interpret the derivative geometrically.

f. The suggestion to guess the answer to a problem sometimes backfired. For example, the students guessed that $\frac{d}{dx} e^x = x\, e^{x-1}$ and that the derivative of a product was the product of the derivatives. Usually, the students appeared to enjoy comparing their expected answer with the actual answer.

g. A slight shift in notation occasionally made a big difference in problem solving. When first encountering implicit differentiation, the students had great difficulty in finding $\frac{dy}{dx}$ for $x^2 + y^2 = 1$, without solving for y in terms of x . The hint was given to replace y by f(x) in the original equation, and the students were then able to find f'(x). The notational shift also clarified the meaning of the functions as representing either the top or the bottom semi-circle.

h. There were occasions when a hint given only once lasted for the remainder of the pilot study. For instance, in the development of the formula for the derivative of a product, the hint was given to add and subtract the same term in the numerator of the difference quotient. In one group it was also necessary to tell the students which term to add and subtract. However, after being given the hint only once, each group of students correctly added and subtracted the appropriate term in every problem where that technique was necessary for the remainder of the year.

i. A very few mathematical suggestions were quite removed from the students' characteristic modes of thought. For instance, in proving the fundamental theorem of calculus, the students found that $F(x) = \int_a^x f(t)dt + C$.

The students knew that C should turn out to be F(a), but they could see no way to obtain that equality. Through probing questions it was found that the students knew the equation held for all values of x in [a,b], but that such information did not appear to be relevant to the problem. The hint was then given to pick a particular value of x that might be useful in achieving the desired identity. In some groups, after several minutes of thought, the students tried putting x = a, but in other groups this suggestion had to be given as an additional hint. Several students remarked, "I couldn't have thought of that in a million years!" Fortunately, there appeared to be very few occasions when the students themselves could not have come up with a particular idea, given sufficient time.

Guiding Suggestions on Group Functioning

The teacher sometimes offered guiding suggestions with respect to the organization of work within a particular group and with respect to the functioning of the group as a social unit. A major suggestion about work organization had to do with the use of the blackboard. The students often wrote four or five attempted solutions to the same problem all over the board, so that it was impossible to tell where one idea ended and the next one began. The students were then hampered by working in a tangle of non-productive, distracting information. In such circumstances the teacher suggested that the students erase all the ideas which didn't work and begin to write in a more systematic manner.

Many students tended to omit writing symbols for the derivative or sometimes for the integral. For example, the students wrote $\sin x = \cos x$, instead of writing $\frac{d}{dx} \sin x = \cos x$. Sometimes students also wrote $\sec^2 x = \tan x$, instead of $\int \sec^2 x \, dx = \tan x$. Such omissions caused no great difficulty on straightforward problems. However, on difficult problems--particularly those involving both differentiation and integration--the omission of key symbols resulted in great confusion. Accordingly, the teacher repeatedly stressed the importance of taking a few extra seconds to write the symbols for the integral and for the derivative, in order to avoid a good deal of confusion, frustration, and reduced efficiency.

Guiding suggestions were sometimes offered with respect to the functioning of a work group as a social unit. These suggestions referred to matters such as cooperation versus competition, conformity pressure, leadership, and dominance by a single individual. These issues are discussed in detail in other sections of the present chapter.

Providing Technical Information

The teacher provided technical information upon request, if the development or recall of that information was not judged to be an important aspect of the problem at hand. For example, a request was always honored for a decimal approximation of the number e to five decimal places. On the other hand, a request was never honored to provide the formulas for $\frac{d}{dx} \tan x$ or for $\frac{d}{dx} f(x)^n$. The students were told that these were important formulas of the calculus, and that students should be able to develop or remember those formulas at any time. Other items of technical information were provided at some times but not at others. For example, the identity $\cos 2\theta = 2 \cos^2\theta - 1$ was given to the students if the use of the identity was a minor aspect of a difficult problem. However, the same identity was not provided if the problem involved computing the integral $\int \cos^2 \theta \, d\theta$. A few students requested very little technical information because they kept a list of important definitions and formulas.

Self-Direction and Error Detection

The teacher stimulated self-direction by encouraging individuals to think through and elaborate upon their ideas. Sometimes this was done by asking the student to explain his ideas to the other group members. At other times the teacher listened to the student's ideas and made brief comments indicating his interest. Examples of such comments are the following: "So you say it's continuous?" "How did you think of that?" "What do you do next?" "Tell us a little more about it." "You said you ran into difficulty at a certain point."

The teacher attempted to stimulate self-direction by encouraging people to detect errors made by themselves or by other group members. It was impossible to judge how many errors were actually detected by the students, but a large number of errors were detected by the teacher. The detection of errors turned out to be one of the major functions of the teacher in supervision of the work groups. The students made numerous errors, and these were patiently pointed out by the teacher without the use of any personal criticism. A few examples are now presented from the voluminous catalogue of student mistakes observed during the pilot study.

a. Errors frequently occurred in computations involving derivatives, integrals, high school algebra, and grade school arithmetic.

b. There were incorrect applications of basic formulas. Students frequently stated that $\frac{d}{dx} \sin 3x = \cos 3x$ because $\frac{d}{dx} \sin x = \cos x$.

c. There were errors showing a misunderstanding of basic algebraic facts. One group stated that $\lim\limits_{n\to\infty} \frac{n^2+7n}{n^3+8n^2} = 0$ because $\lim\limits_{n\to\infty} \frac{n^2}{n^3} + \lim\limits_{n\to\infty} \frac{7n}{8n^2} = 0$.

d. Logical errors frequently occurred in proofs. In doing ϵ-δ proofs about limits of functions, the students often assumed what they wanted to prove, arrived at a true statement such as 2 = 2, and decided that they were finished.

e. There were logical errors of omission in proofs. In many theorems involving a hypothesis about continuity of a function, the students managed to "prove" the conclusion without using the hypothesis.

f. There were errors of over-generalization. Since the students knew that $\frac{d}{dx} x^n = n\, x^{n-1}$, they assumed repeatedly that $\frac{d}{dx} e^x = x\, e^{x-1}$.

g. There were errors of notation, such as:

$f(x) = x^3$ so $f'(x^3) = 3x^2$. Probably none of the errors cited would surprise an experienced teacher of calculus. However, it was surprising to note the frequency with which such errors were made by students with a previous history of better than average achievement in mathematics. Even more surprising were the frequent shifts from moments of ineptitude to moments of billiance on the part of the students. It seemed incredible that students who made so many serious errors could also have so many bright ideas. Some of those bright ideas are reported in Chapter V.

Checking Solutions to Problems

When a group finished solving a problem, the teacher frequently checked the solution. However, the group members sometimes preferred to erase their work and go on to the next problem, without any checking, if the teacher was busy with another group. Checking preferences varied considerably among the groups. Members of some groups always wanted their solution checked and waited until that was done before starting the next problem. Certain other people were generally very confident about the correctness of their solutions and never waited for checking unless the problem was very important or unless there was some disagreement within the group about the correctness of the solution. When enough board space was available, groups left one solution for checking and simultaneously worked on another problem. The previous variations occurred most frequently in computational problems involving limits, derivatives, integrals or their applications. Proofs of theorems were almost always checked, partly because the students were not sure of their proofs and partly because the students were in no hurry to rush ahead and tackle the proof of another difficult theorem.

Providing Extra Problems

Although all the groups began each new topic on the same day, certain groups moved more quickly than others. For example, one group took slightly more than two class periods to finish a set of integration problems, while the other groups required a full three days on the set. The faster group was then given some extra problems of a rather challenging nature. The group had the option of working on the extra problems or of leaving early, and in most instances--until the occurrence of "spring fever"--the faster group stayed to attack the extra problems.

Friendly Relationship with the Students

The teacher felt free to develop a friendly, social relationship with the students. He often arrived a few minutes before class to socialize and to talk about campus happenings, world events, or other topics. On a small number of occasions, the class voted to spend some time discussing non-business matters, such as the presence of police on campus to quell an antiwar demonstration. The small number of out-of-field discussions occurred only when the students were sufficiently excited or disturbed about some campus event to vote as a group to temporarily postpone discussion of mathematics. In addition, some of the students wanted to spend time discussing their personal needs or problems with the teacher; these conversations were sometimes initiated during the class period but were otherwise carried on outside of class time.

Equalitarian Behavior

The teacher behaved in an equalitarian manner by dressing informally, by calling the students by their first names, and by encouraging the students to use his first name if they could feel comfortable in doing so. There was some difficulty in the first-name basis at the beginning of the year. The attitude of some of the students was summed up in the following comment, which occurred during a group discussion in the first week of the class. "What are we supposed to call you? Professor Neil or Doctor Neil or Mister Neil or what?" During the same discussion more than half of the students indicated that they were used to thinking of any teacher as an authority figure and therefore did not feel comfortable with a first-name basis. However, this attitude was scoffed at by some of the other class members, who said that "a teacher can be a human being, just like anybody else." Although the issue was not resolved by the group discussion, the students had a chance to express their doubts and misgivings about being abandoned by the authority figure, and the difficulties with the first-name basis dissolved during the first two or three weeks of class.

The teacher tried to reduce the gap in status between the students and himself by encouraging the students to question or disagree with any of his explanations. He showed little concern about maintaining his dignity, freely admitted not knowing the answers to certain questions, joked about his own mistakes, and refrained from mentioning any high-powered theorems which were beyond the students' grasp at the time.

Preparation of the Notes

In accordance with the plan in Chapter II, there was no textbook for the small group-discovery class. Instead of a textbook there were dittoed notes containing a record of the mathematical content of the course. The dittoed notes relieved the students of the necessity of taking their own notes during class. The dittoed notes were prepared by the teacher after each class meeting in order to include a record of the students' accomplishments on that day.

The notes for each class period were distributed at the following class meeting. The one-day delay in passing out notes was unfortunate, but it seemed to be unavoidable during the first trial of the new instructional method. The teacher usually had expectations about the amount and type of material to be covered on any given day, but these expectations were wrong more often than not. The students frequently encountered unexpected difficulties, sometimes came up with novel problem solutions, and sometimes pursued questions not planned by the teacher. Since the notes were intended to reflect the actual events which transpired on any given day, the one-day delay was necessary. It should be added that the preparation of the dittoed notes was a time-consuming chore.

4.4 COOPERATION VERSUS COMPETITION

The teacher, as supervisor of the work groups, fostered cooperation and discouraged competition through verbal interaction with the group members, emphasizing such matters as the need for joint efforts to solve difficult problems, the importance of listening carefully to the ideas of others, and the goal of solving the problem in such a way that everybody understands the solution. There were plenty of opportunities which arose naturally in the work groups for the teacher to make comments in favor of cooperation. There follows a number of examples of situations and comments by the teacher.

1. Situation: At the beginning of the year, the students had very little conception of the nature of group problem solving. Instead of work groups, there were collections of individuals, each of whom worked by himself in occasional consultation with others.

Comment: "The idea of a group is to have one problem solution agreed upon by everyone, not four individual solutions."

Comment: "Try to build on each other's ideas. If you give part of the solution and somebody else contributes another part, then pretty soon you'll have it all solved."

2. Situation: A student says, "Well, we finally got that problem done, but I couldn't have done it all myself."

Comment: "That's why we have groups. Some of these problems are very hard, and you really have to work together to solve them."

3. Situation: The teacher pointed out an error in a problem solution, and a group member said, " ---, that mistake was your fault!"

Comment: "There is no need to blame anybody for the mistake; this isn't a contest."

4. Situation: For certain problems there were a variety of possible paths to the solution. For example, in evaluating $\int x\sqrt{1-x^2}\,dx$, the four members of one group suggested the following approaches: (1) let $x = \sin\theta$, (2) let $x = \cos\theta$, (3) let $u = 1 - x^2$, (4) write down the answer by inspection. The group members spent more than ten minutes arguing about which approach to take, without reaching agreement.

Comment: "Let's agree to just pick one approach to start the problem. Then, whether it works or not, you can still try out those other ideas afterwards. It might be interesting to see how many different ways you can solve the problem."

5. Situation: One student stated that a particular function was increasing. Someone else said, "No it isn't, because as x gets smaller, the function values get smaller." An argument ensued.

Comment: "How about listening to each other a little more closely. Are you really disagreeing, or just saying the same thing in different words?"

6. In evaluating $\int \tan x \sec^2 x \, dx$, two students obtained different answers, namely $\frac{1}{2} \sec^2 x + C$ and $\frac{1}{2} \tan^2 x + C$. Each student insisted that he was right and the other student was wrong.

Comment: "For one person to be right, must someone else be wrong? Is there any possibility that you're both right?"

7. Situation: Two students were intensely attacking a problem, while two others stood aside for several minutes looking very confused.

Comment: "The group goal is not only to solve the problem, but to solve the problem in such a way that everybody understands the solution."

Comment: "You can feel free to ask questions at any time. If somebody knows what's going on, he'll explain it as clearly as he can."

8. In a rather jealous tone, a student said, "It was his idea that made it work, not mine."

Comment: "But if one person has a good idea, it helps the whole group. Now everybody knows how to do that problem."

Despite the efforts to promote cooperation, there was quite a bit of competition in certain work groups. It was interesting to note the contrast between the different groups. In one group, for example, a rather small number of ideas were produced in problem solving but the atmosphere was quite cooperative. In a second group, numerous ideas were generated but the atmosphere was quite competitive. Although one might expect the group which generated more ideas to make more rapid progress, this was not the case and the two groups moved at about the same rate of speed. It was very noticeable in the competitive group that each member concentrated on defending his own proposed method of solution and did not listen seriously to the proposals of others. Although many ideas were generated, the members had great difficulty in organizing and integrating the ideas into a group solution.

One of the difficulties in promoting cooperation was the presence in the class of three and possibly more individuals who were "firm believers" in competition. One student said that competition "makes people pay more attention and think harder, for the satisfaction of being the first to understand" (Q 28).* Another individual strongly competed with others by proposing numerous ideas for the problem solution and by arguing to make sure that those ideas comprised a large portion of the final solution. Judging from his behavior, he competed in order to impress the teacher, apparently believing that his contributions legitimized his acting as group spokesman to the teacher and also legitimized his attempts to hold rather extended conversations with the teacher about his other mathematical ideas. This person appeared to be puzzled and disappointed when the teacher sought to include other group members in the discussion and suggested that extended conversations during class should be carried on with his colleagues in the group. Hence, for this individual, the non-recognition and non-reward of his competitive prowess seemed to result in frustration, disappointment, and decreased motivation. A private conference with this person would have been desirable, but the teacher did not clearly perceive the problem until the pilot study was over.

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*Any time a reference of the form (Q xx) appears, it indicates a student response to item xx of the questionnaire (Appendix D).

Two practitioners of competition were sometimes in the same small group; the two individuals then carried on a very intense discussion, proposing and defending their own ideas and vigorously attacking the ideas of the other person. This state of affairs tended to exclude the other two group members from the discussion. The other two members were frozen into rather passive roles, and appeared to have quite a bit of difficulty in trying to understand the ideas of the two competitors. The difficulty in understanding was increased by the fact that the two competitors were mainly interested in arguing with each other and were not at all patient in answering questions from others. Once the pattern became clear, the teacher asked the two individuals to work in separate groups for the remainder of the year.

Roughly two-thirds of the class members appeared to be receptive to the practice of cooperation. Although some of these people said they had previously been used to competition, they were willing to try to change their behavior and cooperate with others. One person, who competed at the beginning of the class, said that "I did not enjoy class as much and even concentrated less on the math and more on others' reactions to me" (Q 27). A few people showed some affective leadership in furthering cooperation; they sometimes made statements such as, "Come on you guys, let's work this one out together."

A few individuals, who were normally cooperative, competed in certain groups. This was particularly noticeable in the case of two boys, who both liked the same girl. When in the presence of this girl, they competed mathematically with her and with each other, and she competed with them. The competition usually took the form of verbal put-downs of the other person's suggestions, and criticism of his or her intellectual capabilities. At the point when the situation became bothersome, there was a change in group membership, which resolved the problem without any intervention by the teacher.

In summary, the teacher fostered cooperation through verbal interaction with members of the work groups. About two-thirds of the class members were receptive to the practice of cooperation, and a few of these people showed affective leadership in promoting cooperation in their work groups. About one-third of the class members strongly believed in and practiced competition, and the practice of competition had detrimental effects on other group members when two strong competitors were in the same group. Hence, the teacher asked strongly competitive individuals to work in different groups. All in all, the attempt to further cooperation was partly but not completely successful, and less successful than the teacher had expected. In retrospect, one or more periods should have been devoted to class discussions of the nature of cooperation and of competition.

Competition Between Groups

The work groups usually functioned as separate units, with very little communication between the members of different groups. There appeared to be very few times when members of one group borrowed ideas from the solutions of other groups. There were numerous occasions on which a work group struggled independently to solve a problem, when no more than six feet away, the solution to the problem was written on the board by another group. Nevertheless, many class members did seem to be aware of the progress of other groups relative to their own. When there was a numbered list of problems, people sometimes compared the number, but not the solution, of the problem being solved in their group with the number of the problem being solved in another group. Certain group members became discouraged or inspirited, depending on whether their group was ahead or behind the other groups. Overt instances of competition between groups occurred spontaneously but very rarely and only when there was a numbered list of rather easy problems. On these occasions the competition between groups occurred in the spirit of a lively and enjoyable game, which no one took too seriously and which seemed to entail no loss in understanding of the mathematics. Hence, competition between groups was never a problem and was occasionally an enlivening force.

4.5 LEADERSHIP BY STUDENTS

In accordance with the original plan, the work groups operated without designated task leaders during the entire pilot study. As expected, some group members were much more active and influential than others. To illustrate the two extremes, it suffices to describe the behavior of two individuals. For the entire year, one girl emerged quite clearly as the task leader on the majority of problems, no matter who else was in her work group. Although other group members sometimes competed with her, no one else was able to match her consistent quickness and enormous enthusiasm. At the other extreme was one boy who was the least active and the least influential group member, no matter who else was in his work group. There follows a brief description of the behavior and attitudes of the two individuals.

The Case of Rita

In this paper the name "Rita" is given to the girl who emerged as task leader on the majority of problems, no matter who else was in her group. Some of her attitudes are best described by her responses on the open-ended questionnaire.

"I've always loved it [math] and always will" (Q 85).

"I think it [competition] makes people pay more attention and think harder, for the satisfaction of being the first to understand" (Q 28).

"I'm afraid one of my worst qualities is refusing to be left out" (Q 24).

"Yes, I felt involved when others wrote, but I like writing myself better. I can express my ideas better and faster" (Q 31).

" . . . Things go faster if one person organizes other's comments and thoughts" (Q 33).

"I never look at textbooks anyway" (Q 62).

Certain people enjoyed working with Rita. "I liked it like this because this person inspired some ideas but was not much smarter than us, so we all contributed" (Q 33). Other class members avoided working with Rita. "I was never in that group. I would not prefer that group because it wouldn't offer any challenge to me" (Q 33). As the year progressed, it appeared that people joined her group with full awareness of the nature of the group process involved. One boy, who frequently worked with Rita, served to put some brakes upon her tremendous speed and to make her explain what she was doing. He often sat down, aggressively pointed at her work on the blackboard, and said, "Alright, now explain that to me!" Another boy alternated group membership when a change occurred--one group with Rita, the next group without Rita. He was once heard to say, "I take a lot more responsibility in the group when I'm not with Rita, but I also like working with her because she is so enthusiastic and has so many ideas."

There were interaction difficulties in Rita's groups at the beginning of the year before the students were well acquainted, since many class members had a strong aversion to working in a group largely dominated by a single individual. However, these difficulties disappeared as people became selective in their choice of members of the work groups.

Despite Rita's belief in competition, she appeared to compete in order to be able to explain the solution to the other group members. She never put anyone down, and her enthusiasm was contagious. Moreover, she cheerfully abided by the use of a rotation system which gave the other group members the chance to exercise more leadership. She clearly enjoyed working with other students: "I like working together; when you work alone you can get stuck and disgusted" (Q 76).

The Case of Larry

In this paper the name "Larry" is given to the boy who was the least active and the most quiet member of the class. Larry very rarely made any mathematical suggestions or wrote any solutions on the blackboard. However, on the few occasions when he did contribute an idea, it was almost always correct. In any group containing one or more aggressive students, Larry was a peripheral member, and this was sometimes manifested by his standing several feet away from the other group members. In groups having no very forcible members, Larry appeared to feel more comfortable. He stood closer to the other members, spoke up a bit more, and seemed to be accepted by the others as a part of the group. Upon occasion he solved an entire problem by himself, without talking to the other group members.

At one point during the year, the teacher became worried about Larry's low level of participation and about his possible feelings of being excluded from his work group. In a conference the teacher told Larry that he could feel free to participate as much or as little as he wanted to, but that certain group techniques were available that could help him to speak up more often, if that were his desire. Larry told the teacher that he was really not very concerned about his level of participation or the particular set of members with whom he worked, so long as he learned the math. The teacher decided to push the issue no further. Sometimes, in working with Larry's group, the teacher asked him a question about the mathematics. Larry's reply showed that he did understand the subject matter at least as well and often better than the other group members, and this impression was confirmed by Larry's performance on the take-home examinations.

Participation, Leadership and Dominance

The other group members fell between the two extremes exemplified by Larry and by Rita. A little more than half of the class members were usually very active participants in problem solving throughout the entire year. The other students participated, but not to the same extent. It was very noticeable that certain individuals participated to a greater or a lesser extent, depending upon their particular work group. A person who played a rather subordinate role in one group might exercise a substantial amount of leadership in another group, depending upon the presence or absence of the more aggressive group members.

The differences in power among the group members can be illustrated by recalling two incidents. A striking example of deferential, approval-seeking behavior was shown on one occasion when one of the less active group members worked an entire problem by himself. He then turned to a very influential group member and asked, "Is that right?" The second kind of incident occurred rather frequently. An aggressive group member was unsuccessfully trying to solve the problem and a less influential person had the basic idea for the solution. The person with the idea was neither very direct nor very forcible in his influence attempts, and his suggestion was either not heard or not accepted. The teacher came over to the group and made the very same suggestion, and the person with the idea blurted out, "See, that's what I was trying to tell you five minutes ago!"

In the calculus class, certain people always seemed ready to attack new types of problems, apparently without minding the risk of failure. Other people preferred to watch these attempts and showed leadership only when they were sure that they could succeed. If there were several problems involving the same technique for solution, one person in particular tended to sit back, watch the others struggle, and ask them to explain their attempted solutions. After he gained a feeling of sufficient sureness concerning the new technique, he liked to work an entire problem by himself and simultaneously explain it to the other group members. The other members were usually patient with his attempts, at least whenever the attempts were successful.

After the first few weeks of class, it became clear that certain people tended to dominate the discussion in their work groups. The dominance sometimes resulted in faster progress, but frequently the dominating individual had no better ideas than anyone else and took the group off on a non-productive approach to the solution. It was also observed that the students were extremely reluctant to criticize the group dominators, even when their behavior was clearly detrimental to the progress of the group.

To counter the dominance of certain individuals and to give others more opportunity to participate, three approaches were attempted. (1) The teacher coined the term "chalk hog"; by definition a chalk hog was a person who maintained control of the chalk and who thereby prevented others from writing down their ideas on the board. (Certain groups chose to operate with only one piece of chalk, to reduce disorganization.) It was hoped that students would, when necessary, accuse someone of being a chalk hog and thereby reduce his control over the group. The students did feel free to use the rather humorous term, even though they were reluctant to make more direct criticisms of the behavior of others. Hence, the use of the not-quite-serious "chalk hog" terminology had some effect in reducing the dominance of particular individuals. (2) The second approach was a system of rotation in which different members took primary responsibility for writing the solutions to different problems. This approach worked well when there were numbered lists of problems, but did not work otherwise. (3) The third approach was a system of "experts" on particular homework problems.

The teacher suggested having "experts" on particular homework problems, both to counter the dominance of certain individuals and to guarantee that someone had attempted to work certain difficult problems before coming to class. On particularly difficult problems, there was an expert chosen by each group. The expert was to solve the problem outside of class and to be ready to present his solution to the group when the problem came up for discussion. The expert system did have the desired effect of facilitating contributions by some of the less aggressive group members, and a few class members liked the system. On the other hand, the problems were often too difficult for the supposed experts to solve by themselves. On several occasions the experts came to class with erroneous interpretations of the problem or with quite incorrect attempts at the solution. To avoid this embarrassment, certain experts called up the teacher in order to get some hints on their problems. Moreover, many of the class members did not like the format whereby one person presented a solution to the rest of his work group, since the expert was often anxious and since the other members had little chance to participate. It also turned out that the experts would only work on one problem and would neglect the other problems. Only two or three students liked the expert system, and, as one student put it, "I prefer to work and solve a problem with the other members so it gives us all a sense of satisfaction" (Q 35). The expert system was not included in the original plan for the small group-discovery method, and the system was abandoned after a trial of less than three weeks.

4.6 CONFORMITY PRESSURE

As expected, conformity pressure in the work groups was a danger to be reckoned with. On a number of occasions, the teacher heard a person say, "Well, I suppose we've solved the problem, but I don't see why it works." Another student replied, "Never mind why it works, it just does. Let's go on to the next problem." On such occasions the teacher went over to the particular work group and made a remark such as, "It looks like you've solved the problem, but let's check it out and make sure that everybody understands the solution." Usually, the person who was bothered then stated that he did not understand certain aspects of the solution or that he could follow all the steps without seeing how they fitted together into the final result. The difficulty was then cleared up by means of an explanation from another group member or from the teacher.

Certain class members attempted to apply conformity pressure in the presence of the teacher. For example, the teacher sometimes examined a group solution and said, "That looks pretty convincing to me. Are all of you convinced that it's right?" One group member began to voice a doubt or an objection to the solution, and the doubt or objection was immediately cut off by another student, who said, "Never mind, if he's convinced that it's right, then it's right. We don't need any more questions about this one." The teacher then made a remark such as, "The problem is finished when everybody understands the solution and not before then."

On numerous occasions certain group members displayed expressions of puzzlement or confusion but did not request any information that might clear up their confusion. The teacher then entered the group with a remark such as, "Hi, I just wanted to make sure that everybody understands what's happening." The confused person then said, "I don't, the thing that's bothering me is . . . ." The difficulty was then cleared up. In addition, the teacher tried to stimulate self-direction in asking questions by stating, "You should feel free to ask questions at any time. Most of the time your group members can answer questions without waiting for me to do it."

The incidents described so far have involved either reluctance to ask questions or the application of conformity pressure by a single group member. There were only a few observed incidents in which three group members put pressure upon a single dissenting member. In one such incident, three group members wanted to compute the area under a curve by making approximations with the areas of rectangles. The single dissenting member insisted that it was better to use little squares (as in the double integral). The argument went on rather intensely for a few minutes, and the dissenting member became visibly upset. The teacher then intervened and attempted to lower the tension level by pointing out that both sides were right, but in different ways. The majority was right, as shown by several previous problems, in claiming that it was computationally easier to use rectangles than to use little squares. The dissenter was right in claiming that the use of little squares theoretically gave the same answer as the use of rectangles. The teacher supported the majority decision to use rectangles for computational ease, but also pointed out that the dissenter had a valid approach which would be studied in more detail during the third semester course.

In summary, it was not sufficient to hold abstract discussions about the importance of independent judgment, the legitimacy of disagreement, and the obligation of group members to give reasons to support their statements. During the entire pilot study, there were instances of conformity pressure which necessitated the intervention of the teacher in particular work groups. In addition to overt applications of conformity pressure, there were occasions when students either refrained from asking questions or refrained from stating their disagreement with the views of others until the teacher appeared in the work group.

It should be emphasized that conformity pressure was not a major difficulty during the pilot study. For the most part, the students did ask questions, did try to answer the questions of others, did disagree with others when necessary, did try to give reasons to support their statements, and did not pretend to understand what was unclear. Instances of conformity pressure appeared recurrently, but these instances were the exception, rather than the rule, in the functioning of the work groups.

4.7 GROUP SIZE

During the pilot study there were four members in each work group. It appeared that the 4-member work group functioned rather well on theoretical problems. The students were not in general very confident of their ability to prove theorems, and contributions by all the group members seemed to be useful in generating enough ideas to do the proof. However, the 4-member group sometimes appeared to be too large for the optimal solution of standard computational exercises. In working computational problems, the 4-person group sometimes split spontaneously into 2 subgroups with 2 members apiece. The 2 subgroups worked independently of each other but compared their answers to the problems. When the computational problems were not very difficult, the splitting into subgroups gave each person the opportunity to get more practice than in the 4-member group. When the computational problems were not difficult and when the 4-member group did not split up, certain class members appeared to be rather bored.

When one group member was absent, the remaining three members were able to function without any great loss in efficiency except in two circumstances. Efficiency was reduced when the problems were very difficult or when the missing member normally exercised a great deal of leadership in the group. There were a very few occasions when there were personality clashes in a group and when the absence of one of the clashing members seemed to improve group efficiency and morale.

On the very rare occasions when two members of a work group were absent, there were alternate modes of operation. If the two members remained together for the day, they limped along with reduced efficiency and morale and required more than the usual amount of teacher guidance. The two members sometimes chose to split up for the day and to join different groups. When an extra person joined an existing group, the effects were not beneficial. The extra person either stood aside as a peripheral member or else participated in an overly aggressive manner and disrupted the functioning of the existing group.

4.8 SUMMARY

In summary, the design reported in Chapter III was carried out during the pilot study. There were a few instances, which have been noted, in which the design did not quite work in the expected manner. In addition to the decisions pertaining to the classroom organization and social climate, there were some plans regarding the treatment of the mathematics content. The interaction between the students and the mathematics content is the subject of the next chapter.

Chapter V

THE INTERACTION BETWEEN THE STUDENTS AND THE MATHEMATICS CONTENT

5.1 OUTLINE OF CHAPTER V

The present chapter is an abbreviated anecdotal record of the interaction between the students and the mathematics content during the pilot study. The present chapter, unlike the previous one, does not include a description of the implementation of a set of general procedures. The main body of the chapter begins with an outline of the content covered during the pilot study. A discussion follows pertaining to the amount of material covered and the method of organization of the material.

Since the teacher had expected the students in their groups to prove the major theorems and to develop techniques for problem solving, there is limited emphasis in the present chapter upon the student success in doing so. Instead, some information is presented regarding the most surprising affective and cognitive aspects of the student interaction with the content. There is also a brief account of the manner in which this information was obtained. Although some of the information pertains to student accomplishments, much of it pertains to student difficulties and attitudes toward the content. The particular issues include (1) cognitive processes, (2) topics in differentiation and integration, (3) the definition of the limit, (4) functions, differentiability, and continuity, and (5) theory versus problems.

5.2 CONTENT OUTLINE FOR THE PILOT STUDY

Semester I

Topic			Number of Days
Unit	I.	Preliminary Review	8
	1.	Roots and factors of polynomials, completing the square	1
	2.	Logarithms	1
	3.	Trigonometry	2
	4.	Principle of mathematical induction	2
	5.	Binomial theorem, Pascal's triangle, number of subsets of a finite set	2
Unit	II.	Limits of Sequences	24
	1.	Area of a circle, arrow paradox	1
	2.	Geometric series	1
	3.	Convergent sequences and divergent sequences	4
	4.	Rational functions	1
	5.	Repeating decimals, polynomials	1
	6.	Areas as limits of sequences	2
	7.	Sums of sequences	1
	8.	Sequences defined by factorials or trig functions	1

9.	Infinite series	1
10.	The number "e"	1
11.	Product sequences and reciprocals	2
12.	Definition of the limit: formulation	2
13.	Definition of the limit: application to particular sequences	4
14.	Definition of the limit: proofs of some limit theorems	2
Unit III.	Limits and Derivatives of Functions	17
1.	Graphs	1
2.	Limits of functions: computations	1
3.	Tangent lines and derivatives	2
4.	Derivatives: computations with the definition	2
5.	Derivatives of the sine and cosine functions	2
6.	Derivative of the absolute value function, statement of limit theorems	1
7.	Derivative formulas, alternate notations for the derivative	2
8.	Derivatives: computations using formulas	2
9.	Derivatives of trig functions	1
10.	The chain rule: proof	1
11.	The chain rule: computations	2
Unit IV.	Properties of Functions and their Graphs	11
1.	Intervals, domain of a function, continuity	1
2.	Discontinuous functions, relation between continuity and differentiability	1
3.	Relative maxima and minima	1
4.	Absolute maxima and minima, Rolle's theorem	1
5.	Mean value theorem	1
6.	Applications of the mean value theorem, functions having zero derivatives	1
7.	Increasing functions and decreasing functions	1

	8.	Curve tracing with the first derivative	1
	9.	Concavity, inflection points	1
	10.	Curve tracing using two derivatives	1
	11.	Curves with asymptotes	1
Unit	V.	The Definite Integral	7
	1.	Upper and lower sums, definition of the integral	1
	2.	Non-integrable functions	1
	3.	Relationship between integrals and areas	1
	4.	Bounds on definite integrals, the average value theorem	1
	5.	Conjecture of the fundamental theorem, use of the fundamental theorem	1
	6.	The derivative of an integral (lemma for the fundamental theorem)	1
	7.	The integral of a derivative (the fundamental theorem)	1

Semester II

Unit	VI.	Further Uses and Pr[illegible]erties of Derivatives and Limits	15
	1.	Motion on a straight line: velocity and acceleration	1
	2.	Implicit differentiation	1
	3.	Differentials and their use in approximations	1
	4.	Equations of motion, simple differential equations	3
	5.	Rate of change problems	2
	6.	Maxima and minima problems	2
	7.	Definition of the limit of a function: formulation and geometric interpretation	2
	8.	Definition of the limit: application to polynomials	2
	9.	Definition of the limit: theorems	1

Unit VII.	The Logarithm, Exponential, and Inverse Trig Functions	14
1.	Natural logarithm and exponential functions: algebraic properties	1
2.	Derivatives of the natural logarithm and exponential functions (informal development)	1
3.	Natural logarithm and exponential functions: indefinite integrals	3
4.	Definition of the natural logarithm as an integral: basic properties	2
5.	Definition of the exponential function to various bases: algebraic properties, derivatives, indefinite integrals	3
6.	Logarithmic differentiation	1
7.	Inverse trig functions: definitions, graphs, identities, derivatives, and some indefinite integrals	3
Unit VIII.	Integration Techniques	20
1.	Integration by inspection or substitution	1
2.	Integration of certain powers of trig functions	3
3.	Integration of even powers of sines and cosines	1
4.	Integration by trig substitutions	4
5.	Integration by completing the square	1
6.	Integration by partial fractions	4
7.	Integration by parts	3
8.	Integration by parts with reduction formulas	1
9.	Improper integrals	2
Unit IX.	Analytic Geometry	11
1.	Perpendicular lines, the circle	1
2.	The parabola	2
3.	The ellipse	2
4.	Line and point symmetry	1
5.	The hyperbola	3
6.	Rotation of axes, the discriminant, the general second degree equation	2

Unit	X.	Parametric Equations and Vectors	4
	1.	Parametric equations	1
	2.	Two-dimensional vectors, components, length, unit vectors	1
	3.	Vector sum and derivative	1
	4.	Tangent and normal vectors	1
Unit	XI.	Applications of Integration	5
	1.	Areas under and between curves	1
	2.	Volumes by the cylinder method	1
	3.	Volumes by the shell method	1
	4.	Arc lengths of curves	2
Unit	XII.	Polar coordinates	3
	1.	Coordinates and curves	2
	2.	Area in polar coordinates	1

5.3 TOPICS OMITTED DURING THE PILOT STUDY

It seems appropriate at this point to discuss material which was not covered during the pilot study. The preliminary review and the five-week unit on sequences necessitated the omission of certain topics which were covered in one of the lecture classes at the University of Wisconsin. The topics covered by that lecture class and omitted by the small group-discovery class were the following:

1. Numerical integration with the trapezoidal rule and Simpson's rule
2. One substitution technique for evaluating integrals
3. Certain applications of integration: surface area, fluid pressure, work, moments, the centroid
4. Proof of the existence of the integral of a continuous function on a closed interval

5. Newton's method
6. Determinants and linear equations (a unit)
7. Three-dimensional analytic geometry and vectors (a unit)
8. Curvature
9. Proofs of two limit theorems pertaining to the limit of a product and of a quotient

It should also be added that the units on parametric equations and vectors and on polar coordinates were handled in a rather sketchy form in the small group-discovery class. Finally, among the items not covered in the small group-discovery class, only two were omitted on grounds of their inherent difficulty--the proof of the integrability of a continuous function and the proofs of two limit theorems. The other items were omitted for lack of time.

5.4 ORGANIZATION OF THE SUBJECT MATTER

In accordance with the plan in Chapter II, the teacher tried to organize the subject matter with concrete experience first and abstraction later. This point can be illustrated with several examples. The definition of the limit of a sequence was placed at the end of a five-week unit on sequences. The definition of the limit of a real-valued function was postponed until the second semester, by which time the students had spent several months working with limits, derivatives, and integrals of functions. The definition of the definite integral in terms of Riemann sums was deferred until the end of the first semester. However, concrete experience with the definite integrals representing certain areas occurred at a very early stage, without the use of any formal terminology. The definition of the natural logarithm function as a definite integral took place after an informal development of the derivative of the natural logarithm function, using the fact that $e = \lim_{n \to \infty} (1+1/n)^n$.

The intention to provide concrete experience before abstraction was not carried out to the fullest extent. For example, the difficult proof of the fundamental theorem of calculus could have been postponed until the end of the unit on integration techniques. The applications of the definite integral in computing areas, volumes, and arc lengths could have preceded or dovetailed with the unit on integration techniques. The decision to leave the applications of the integral until late in the second semester was made solely on the grounds of time pressure. Finally, the proofs of certain theorems about curves could have been postponed, pending the occurrence of a good deal of concrete experience in sketching particular curves. In short, with regard to the organization of subject matter, the behavior of the teacher coincided partially with his stated intentions.

5.5 OBSERVATIONS OF THE STUDENTS AT WORK

Through daily conversations with the students and through daily observations of their work, the teacher acquired a substantial amount of information about the interaction between the students and the subject matter. Some of the information was of a mathematical nature, ranging from false assumptions and computational errors to clever solutions and insightful conjectures. Some of the information was of an affective nature, since certain students spontaneously displayed their reactions to the subject matter. Without any solicitation from the teacher, they often made evaluative comments about the material and showed either their enthusiasm or their dislike for it. Some of the sample comments were the following. "Hey, this stuff is really great!" "These problems are much more interesting than that junk we had yesterday!" "I never liked inverse trig functions in high school and I don't like them any better now!" "Oh no, not another theorem!" In addition, the students often made exclamations at a moment of insight. "Aha!" "I see it, I see it!" "Wait, stop, I've got it!"

Throughout the present chapter, there are a number of assertions that the students said, did, wrote, or perceived something. These assertions refer to the students as a group--either a work group or the total class, depending on the situation. No claim is made that the assertions represent the views of every single member of the group. However, in many instances the same phenomenon was observed in all three work groups. Moreover, when exceptions were observed to the general trend, the assertion is presented in a qualified form. There is the danger, of course, that the teacher was overly conscious of the views of the vocal group members and not very conscious of the views of the others. The discussion which follows is meant to be interpreted in the light of these cautionary remarks.

5.6 SOME DIFFICULTIES WITH COGNITIVE PROCESSES

During the pilot study, most of the students had difficulty with the recall of definitions, with the transfer of information from one problem to other problems, and with the testing of universally quantified statements in terms of particular instances. There follows a description of the three types of difficulty.

Non-Use of Specific Instances of Propositions

The students frequently did not test universally quantified statements in terms of particular instances. For example, many students had great difficulty in correctly recalling the identity $1 + \tan^2 \theta = \sec^2 \theta$. Each time the students wrote down an incorrect version of the identity, the teacher asked if there was any easy way to test the truth of their statement. It was usually necessary to tell the students to try their identity with a particular value of θ. Moreover, the students usually treated the identity as something which could only be remembered, rather than as a result which could easily be derived.

The failure to test statements by using specific instances was especially noticeable in trigonometric settings. In addition to the difficulty already cited, there was trouble with the formulas for the sine and the cosine of the sum of two angles. For example, when some students said that $\cos(x+y) = \cos x \cos y + \sin x \sin y$, it was necessary for the teacher to suggest setting $x = y$ in the formula. Moreover, there were numerous instances of incorrect recall of the formulas for the derivatives of trig functions. For example, some students stated that

$\frac{d}{dx}$ sec x = $\tan^2 x$. It did not occur to them to try particular values of x or to sketch the graph of the secant function and see whether its derivative could always be non-negative.

Each Problem Unique unto Itself

There was a noticeable tendency for the students to treat all problems as if they were separate and unrelated to each other. On a number of occasions, the students did not perceive the "obvious" approach of using the result of one problem to solve another problem. For example, the students initially computed derivatives by means of the definition. They then spent several days developing formulas for the derivatives of sums, products, functions to powers, and so forth. Right after this development, the students were given a new set of complicated-looking functions for differentiation. The groups immediately reverted back to using the definition of the derivative, rather than their new general formulas.

In a second example, the groups worked out the formulas for the derivatives of the sine and the cosine functions. In finding the derivative of the tangent function, the groups all set $\tan x = \sin x / \cos x$. Instead of using the quotient rule, the members of one group used the definition of the derivative. With the use of various trig identities and clever manipulations, they managed to carry off a long and complicated development of the derivative of the tangent function.

The non-relating of problems to one another was also observed in the topic of integration techniques. For instance, the groups first evaluated $\int \frac{dx}{a^2 + x^2}$ by means of the substitution $x = a \tan \theta$. Instead of using this result, the groups then evaluated $\int \frac{dx}{9 + x^2}$ by the substitution $x = 3 \tan \theta$. Although a little repetition is sometimes a useful aid in learning, many students repeated the same work over and over again, long after they had mastered the technique of integration by trig substitutions.

Non-Recall of Definitions

The students were apparently not used to thinking in terms of definitions and they tended to forget major definitions from one day to the next or even from one problem to the next. Most of the students were persistently unable to recall major definitions, including the definitions of the limit, the derivative, the definite integral, and continuity.

The definition of the derivative fared slightly better than the other definitions, perhaps because it is a simple formula or perhaps because it was frequently used.

The students often tended not to use the major definitions, even in those problems where the use of the definition provided the only possible approach. For example, the problem was given to determine the integrability or non-integrability of the following function: $f(x) = 0$ if x is rational, $f(x) = 1$ if x is irrational, where the domain of f is $[0,1]$. The students had an exciting discussion in their groups as to whether the function defined a curve and, if so, whether there was an area underneath the curve. The teacher simply listened to the discussion for about ten minutes before asking, "How are you going to tell whether the function is integrable or not?" The reply was, "We're still trying to decide whether there is any area under the curve, and when we know that we'll know if the function is integrable." The teacher had to suggest that the definition of the integral might be related to the existence or non-existence of an area. The students had a difficult time in writing down the definitions of the upper and lower sums and in computing those sums for the given function. With several hints they were finally able to carry out the details and concluded that the function was not integrable.

A second example of the non-use of definitions is provided by the proof of the following theorem: if $f'(c)$ exists and if f has a relative maximum at $x = c$, then $f'(c) = 0$. Some of the students correctly drew a picture with some pertinent secant lines and concluded that the slopes of those lines were positive for $x < c$ and negative for $x > c$. They then stated that the derivative was obviously 0 from the picture. The suggestion was given to prove their statements by translating the picture into appropriate mathematical symbols. The further suggestion was needed that the definition of the derivative might be pertinent. An additional obstacle in the proof was the need to recall the definition of a relative maximum, which the groups had formulated on the previous day.

When the students were working with the definition of the limit, the teacher asked each group to record the definition on the blackboard before working any problems. Although this was done for a number of days in a row, the recall process was never efficient and was usually marked by a surprising variety of mis-statements. Furthermore, although the definition of continuity is a very brief statement, the students almost always had trouble in remembering or recognizing it, apparently because they basically believed that all functions were continuous. The student encounters with continuity and the definition of the limit are described in detail elsewhere in this chapter. There now follows a discussion of the student interaction with three important topics in differentiation and integration.

5.7 THREE TOPICS IN DIFFERENTIATION AND INTEGRATION

The Derivative of a Composite Function

There were persistent difficulties in working with the chain rule for the derivative of a composite function. The main difficulty was that most students could not see the need for applying the rule in new situations. Each time an important new function was introduced, the student groups correctly developed the formula for its derivative. They then made erroneous computations such as $\frac{d}{dx}\sin 2x = \cos 2x$, $\frac{d}{dx}e^{3x} = e^{3x}$, $\frac{d}{dx}\ln 4x = \frac{1}{4x}$, and $\frac{d}{dx}\sin^{-1} 5x = \frac{1}{\sqrt{1-(5x)^2}}$. In each instance it was necessary for the teacher to make statements such as, "Can you see that as a function of a function?" or "How about picking a new variable to use in your formula." In many instances these hints were sufficient, but in other instances it was necessary to ask the students to write down the chain rule and apply it in a step-by-step manner. There were, of course, the usual mechanical difficulties in working with the chain rule, but these difficulties were much less striking than the fact that most students did not even perceive the applicability of the chain rule in new situations.

The Differential

Almost all of the students had persistent difficulties in working with the differential in problems of integration. For example, in evaluating $\int \sin^3 x \cos x \, dx$, the groups let $u = \sin x$. They then tried to evaluate $\int u^3 \cos x \, dx$ without transforming the integrand into a function of only one variable. The teacher suggested that they make use of the relation $u = \sin x$, and they did so in the following incorrect ways: $du = \cos x$, or $\frac{du}{dx} = \cos x \, dx$. It was necessary for the teacher to suggest that the students simply take the derivative in the ordinary manner and then multiply through by the dx term. In many problems with integration by substitution or by parts, the students forgot to make the conversion of the variables in the integrand.

Then, when they tried to do so, they frequently ended up with either too many or too few differential symbols. These difficulties finally decreased at the end of a long unit on integration.

Integration by Trigonometric Substitutions

The technique of integration by trigonometric substitutions was introduced with the problem of computing the area of a circle. After some difficulty in choosing an appropriate function, the groups set up the integral $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$. The group members attempted to compute the integral for three or four minutes, using techniques which were already familiar. The hint was then given to choose a new coordinate system. The groups rather quickly decided to make use of polar coordinates and to set $x = r \cos \theta$, where r was the given radius of the circle. They soon arrived at the integral $\int_{-r}^{r} r \sin \theta \, dx$. The two remaining obstacles were the conversion of the dx term and the limits on x into a $d\theta$ term and limits on θ ; it was necessary for the teacher to state that these changes were needed.

After computing the area of a circle of radius r, the next problem was to evaluate $\int \frac{dx}{\sqrt{r^2 - x^2}}$. Many students believed they had no way to attack the problem. When the suggestion was given to consider the method used for the previous problem, several students stated that the substitution $x = r \cos \theta$ could not be used, since the present integral had nothing to do with a circle. The teacher had to suggest that the same substitution might still work, since the relation $x = r \cos \theta$ restricted only the x-values and not the function values.

It was interesting to note that in other integrals involving the term $r^2 - x^2$ the student always used the substitution $x = r \cos \theta$, rather than the more standard substitution $x = r \sin \theta$. When asked if there was a substitution other than $x = r \cos \theta$, several students said that $r \sin \theta$ would also work but that they liked their first approach better. Later, when first encountering integrals involving the term $r^2 + x^2$, the groups tried using $x = r \cos \theta$ without any success. The hint was given to consider another trig identity besides $1 - \cos^2 \theta$; in each group a student then let $x = r \tan \theta$.

5.8 THE DEFINITION OF THE LIMIT

Certain Puzzling Sequences

Almost all of the students initially believed that the sequence 1, 1, 1, 1, ... was not a sequence at all, since the relation $x_n = 1$ did not change with n and since x_n was not specified by a formula with n as a variable. It took several minutes of argument to resolve this issue, and another surprising event then transpired. The students were almost all convinced that the sequence 1, 1, 1, ... did not have a limit. "To say that 1 is the limit would mean that x_n gets closer and closer to 1 as n gets very big. But it's not getting closer and closer to 1; it's there already." A very few students believed that 1 was the limit, and a rather intense argument took place. The students then appealed to the teacher to settle the issue, and he stated that mathematicians accepted 1 as the limit. Certain students consoled themselves by stating that definitions were arbitrary anyway and that it was only important to be consistent in using the definitions. However, other students resisted all the attempted explanations, stating that the mathematicians' definition was in clear violation to the reasonable intuitive notion of the limit.

The sequences 0,2,0,2,0,2,... and 1, 1/2, 5, 1/3, 1/4, 5, 1/5, 1/6, 5, ... engendered a controversy. In each instance a number of students said that the given sequence converged to two different limits. After an argument the students again appealed to the teacher, who stated the mathematicians' position that a sequence could only have one limit. There was a discussion of the entire class, in which the students debated whether to accept the mathematicians' definitions or to make up their own. As mentioned in the previous chapter, the students finally voted to accept the conventional definitions.

Some students were rather upset by the group decision that the sequence 0,2,0,2,... had no limit. These students were somewhat mollified when the teacher introduced the notion of a subsequence and stated that the given sequence could be split into subsequences with limits 0 and 2 respectively.

The Definition of the Limit of a Sequence

The formulation of the definition of the limit of a sequence was a three-stage process involving the following cases: (1) $\lim_{n\to\infty} x_n = 0$, where $x_n > 0$ for all n, (2) $\lim_{n\to\infty} x_n = 0$, (3) $\lim_{n\to\infty} x_n = L$.

In case (1), with $\lim_{n\to\infty} x_n = 0$ and $x_n > 0$ for all n, the teacher asked the following question of the entire class: "How small does x_n get?" The intent was to begin a discussion which would continue in the work groups, but one student immediately responded, "x_n gets smaller than any fixed positive number." People nodded their heads in agreement, and the following dialogue took place.

Teacher: "Suppose that α is any fixed positive number. Can you translate your statement into symbols?" (The letter "α" was used to avoid associations with "that terrible ϵ" the students had heard about from members of other classes.)

Response: "$x_n < \alpha$."

Teacher: "Is that true for all n?"

Response: No one was able to answer the question.

Teacher: "Let's answer that question in the work groups by considering sequences with general terms $\frac{1}{n}$, $\frac{1}{n^2}$, $\frac{1}{2^n}$, respectively. It will also help to take particular values for α, say $\alpha = \frac{1}{100}$ and $\alpha = \frac{1}{1,000}$."

In the work groups the students found that for each sequence, $x_n < \alpha$ if n is bigger than some appropriate number (for the values $\alpha = \frac{1}{100}$ and $\alpha = \frac{1}{1,000}$). For α arbitrary, the students found that $\frac{1}{n} < \alpha$ if $n > \frac{1}{\alpha}$ and $\frac{1}{n^2} < \alpha$ if $n > \frac{1}{\sqrt{\alpha}}$. There were technical difficulties in solving the inequality $\frac{1}{2^n} < \alpha$, and the discussion of the entire class reconvened briefly.

Teacher: "So what did you find out? Is it true that $x_n < \alpha$ for all n?"

Response: "No, only for some values of n."

Teacher: "Can you say a little more about those values of n?"

Response: "They are all bigger than some function of α . $x_n < \alpha$ if n is bigger than some function of α ."

Teacher: "We've already seen that there are some sequences where it's not easy to actually find that function of α, so let's simplify our statement a little. If α is a fixed number, what is that function of α?"

Response: "Another fixed number."

The result in case (1) was then written as follows: if α is any fixed positive number, $x_n < \alpha$ if n is bigger than some number N.

Cases (2) and (3) were handled by the students in their work groups. In case (2) x_n approaches 0 without any restriction that x_n be positive. The students considered the example $x_n = (-1)^n \cdot \frac{1}{n}$. In each group someone noticed that $|x_n|$ becomes smaller than any fixed positive number as n approaches ∞ . The result was the following: if α is any fixed positive number, $|x_n| < \alpha$ if n is bigger than some number N . In case (3) some students in each group noticed that the distance between x_n and the arbitrary limit L becomes smaller than any fixed positive number. There was a technical difficulty because some students could not remember how to find the distance between two given points on the real line. Nevertheless, the final result emerged as follows: if α is any fixed positive number, $|x_n - L| < \alpha$ if n is bigger than some number N .

After the previous development, the teacher said, "Well, there's an interesting statement. What do you think it's good for?" The students were unable to indicate any possible uses of the statement, and most of them were astounded when the teacher suggested that the statement might serve as the definition of a limit. The student comments expressed their disbelief. "What, that thing a definition?" "You've got to be kidding." "If that's a definition, it's the wierdest one I've seen in my entire life." Rather than trying to make a case for the proposed definition, the teacher suggested that its usefulness might be tested by applying it to some particular sequences, the convergence or divergence of which had already been informally ascertained. Some of the sequences to be considered were (a) $x_n = \frac{1}{n^3}$, (b) $x_n = 1$ for all n, (c) $x_n = 2$ for n even, $x_n = 0$ for n odd. The definition would be rejected if its application contradicted the previous informal results, and the definition might be accepted if it confirmed those results. In the latter eventuality, the utility of the proposed definition might be further tested by attempting to prove some previously unprovable results--for example, that the limit of a sum is the same as the sum of the limits.

Application of the Definition

The students encountered enormous difficulties in working with the definition of the limit of a sequence. Some of the difficulties can be illustrated by the student attempts to prove the following theorem: if $\lim_{n\to\infty} x_n = L$, then $\lim_{n\to\infty} cx_n = cL$. When it became apparent that the students could not initially do the proof with an arbitrary constant c, the suggestion was made to try the cases $c = 3$ and $c = \frac{1}{3}$. A number of students observed that $|3x_n - 3L| < \alpha$ if and only if $|x_n - L| < \frac{\alpha}{3}$, but then claimed that the latter statement was true because $\frac{\alpha}{3} < \alpha$. In working with $c = \frac{1}{3}$, many students observed that $|\frac{1}{3}x_n - \frac{1}{3}L| < \alpha$ if and only if $|x_n - L| < 3\alpha$. They then said they were stuck because $3\alpha > \alpha$. Most students could not seem to understand the distinction between working with a given value of α in order to prove the convergence of a sequence, and choosing a suitable value of α for a sequence already known to be convergent. The suggestion to trace through the proof with particular numerical values of α helped to some extent, and the groups were finally able to construct a correct proof. However, in a number of other problems, there was the same difficulty with being able to choose a convenient value of α at some times but having to stick with a given α at other times.

In proving theorems about limits of sequences, the students were constantly bothered by the presence of literal symbols such as α, N, x_n, and L and by the meaning of the terminology "given any α" and "for some N." Moreover, it was not very helpful to use alternate phrases such as "for every α" and "there exists N." However, it was sometimes helpful in proving theorems to use geometric interpretations on the number line and to first make the argument with particular values of α and N or with particular sequences or both. The use of particular values was time-consuming, but the choice was between slow progress or no progress on the definition of the limit.

It is important to note that the difficulties with the definition of the limit were not caused by lack of concrete experience with sequences. The students had done computational work with numerous specific sequences and had spent ample time arguing about the convergence or divergence of puzzling sequences such as 0,2,0,2,0,2,... . They had combined sequences in various ways and had worked with sequences involving repeating decimals, the number e, areas under certain curves, and infinite series. The definition of the limit came at the end and not the beginning of the unit on sequences.

Limits of Functions in Comparison with Limits of Sequences

As expected, the students found that the definition of the limit was easier for sequences than for functions. In the case of sequences, there was no opportunity for confusion of tiny α and huge N--unlike the case of functions where ϵ and δ are both small. In the case of sequences, n is positive and approaches ∞ from one direction only, whereas in the case of functions, x can approach x_o from either side with possibly different results. Finally, as observed in the pilot study, students were not tempted in the case of sequences to set $n = \infty$. In the case of functions, many students tried to "prove" the result by setting $x = x_o$ and were confused by the restriction $0 < |x-x_o| < \delta$.

Affective Responses

In the design of the small group-discovery method, the intent was to work with only those mathematical topics which students found either interesting or useful. According to students responses on the questionnaire (Appendix D) the topic of the definition of the limit was not interesting. This attitude was observed in class, when several students said, "Oh, here comes another one of those hard, boring limit problems." The proofs were also rather unconvincing. "That proof is nothing but a bunch of equivalent statements with complicated notation. It doesn't prove anything to me." Moreover, most students did not find the proofs of limit theorems to be useful for anything. One girl spoke for a number of other students when she said, "There's no reason to prove a theorem unless there is some doubt about the result, and I never had any doubt about the sum of the limits being the same as the limit of the sum." Several students on different occasions made remarks such as, "You mean that we've spent all this time and all this effort just to prove something that was obvious in the first place?"

In conclusion, most students found the easier sequence version of the definition of the limit to be not interesting, not useful, and extraordinarily difficult. Most students found the function version of the definition to be not interesting, not useful, and even more difficult. An attempt was made to reduce the frustration level in working with the definition of the limit of a function by spending only part of the class period on that topic and by doing other material for the rest of the period. After one week the topic was abandoned, because the students almost invariably found the other material more enjoyable and complained about having to spend any time at all on the definition of the limit.

The student reactions to the definition of a limit are reported here with considerable regret. The pilot study had commenced with the topic of sequences partly in order to permit the students to formulate and work with an easier version of the definition of the limit. It had been hoped and expected that the previous concrete experience, plus the opportunity to formulate and work leisurely with the definition, would render the proofs of limit theorems both meaningful and comprehensible to the students. Although the formulation stage went very well, the application of the definition was a painful experience for many of the students, except in the case of very simple problems.

5.9 FUNCTIONS, DIFFERENTIABILITY, AND CONTINUITY

The Student Concept of Real-Valued Functions

The student concept of real-valued functions seemed to include several basic but unstated assumptions. When drawing the graph of a function defined on a closed interval, the students invariably drew a smooth curve with a small number of relative maxima and minima. Moreover, a student was heard to say, "There are only three possibilities at an endpoint of the interval. Either the curve comes in level or it comes in from below or from above." The other three members of the work group indicated their agreement with that assertion. It appeared that the student concept of a function defined on a closed interval actually meant a continuous, differentiable function with finitely many relative maxima and minima. Further evidence will be cited in support of this contention.

Continuity

The students were almost always unable to state or to recognize the definition of continuity. In many problems students automatically used the property "$\lim_{x \to x_0} f(x) = f(x_0)$," without noticing that the property might not hold for the particular function in question. Moreover, the students tended to automatically assume the existence of absolute maximum and minimum values for any function defined on a closed interval. When asked what property of the function guaranteed the existence of a maximum value of the function, the student reply was usually either a blank stare or a remark such as, "What do you mean; how could it not have a maximum value?"

Almost all of the students reacted very negatively to functions with removable discontinuities--for example, $f(x) = 1$ if $x \neq 0$ and $f(0) = 2$. They seemed to feel that such functions were unreal figments of the mathematician's imagination, and they made remarks such as, "That function is ridiculous. Why don't you just put f(0) back where it belongs!" Many students were also bothered by step functions with non-removable discontinuities. "Where do you ever find a function like that in the real world?" The familiar example of the postage stamp function did not seem to convince the majority of students of the mathematical importance of step functions. Hence, the examples cited so far of discontinuous functions were artificial and "trumped up," as perceived by many of the students.

There were other instances of functions having points of discontinuity. The students were all familiar with the function $f(x) = \frac{1}{x}$ for $x \neq 0$. In order to discuss continuity at 0 , the domain of the function was extended by setting $f(0) = 0$. Although many of the students recognized that the new function was not continuous at 0 , they were not impressed by that fact. "The only reason it's not continuous at 0 is that we added a point which shouldn't have been there in the first place."

A further example of a discontinuous function was given by the rule $f(x) = \sin \frac{1}{x}$ if $x \neq 0$, $f(0) = 0$. The students had not seen this function before, and they had a hard time trying to make a rough sketch of the curve. Moreover, most students were at first convinced that the function had a limit as x approached 0 . Different groups found different limits such as 0, 1/2, and 1 by restricting attention to certain sequences of x values. It took more than 10 minutes of argument before everyone was convinced that $\lim_{x \to 0} f(x)$ did not exist.

In summary, from the perspective of most students there was something unnatural or artificial about the examples of functions having discontinuities. Contrary to the students' previous experience, the functions were not specified by single formulas. Instead, the functions were constructed by moving points out of their proper location or by adding points which did not appear to belong in the domain or by creating infinitely many oscillations in the graph.

Existence of the Derivative

Initially, the students seemed to think that all functions were differentiable. The problem was given to investigate the derivative of the function $f(x) = |x|$. The groups correctly found that $f'(x) = 1$ if $x > 0$ and $f'(x) = -1$ if $x < 0$. They then addressed the case $x = 0$ with the stated intention of finding $f'(0)$, rather than of investigating the existence of the derivative at 0 . When the right-hand and the left-hand limits of the difference quotients turned out to be different, most of the students thought that they had made an arithmetic mistake somewhere. It took a geometric interpretation and several minutes of discussion before all the group members believed that f'(0) did not exist.

During the pilot study the students encountered a number of functions which failed to have a derivative at some point. Among the examples were the functions $h(x) = x^{1/3}$, and $g(x) = \sqrt{1 - x^2}$ for x in [-1, 1]. In the various examples, the groups were able to make sketches and to find vertical tangent lines at the appropriate points. In other examples such as $f(x) = |x|$, the groups were able to show that the right-hand tangent line at 0 did not fit the left-hand portion of the curve, and vice-versa. Nevertheless, in stating theorems about curves, the students sometimes forgot about the existence of functions which were not everywhere differentiable.

Differentiability versus Continuity

Although many students were somewhat bothered by functions which were not everywhere differentiable, they were much more bothered by functions having discontinuities. The teacher interpreted this phenomenon in the following way. There were several examples of functions which were specified by simple formulas but not everywhere differentiable.

However, the examples of discontinuous functions were unfamiliar, artificial, and difficult as perceived by most of the students.

The student aversion to discontinuous functions may explain another phenomenon which occurred persistently throughout the pilot study. The students almost always reversed the relationship between differentiability and continuity and stated that all continuous functions were differentiable. The confusion is perhaps not surprising in light of the student viewpoint that continuous functions were proper and "good" but discontinuous functions were artificial and "bad." A continuous function--that is, a "good" function--should have other "good" properties such as differentiability. Removing the basic flaw, namely the discontinuity, should remove all the other flaws as well. To use a medical analogy, if one cures the basic malady of a sick patient, a number of associated symptoms may also disappear. However, it must be emphasized that this explanation is an interpretation of some rather puzzling behavior and not a report of anything explicitly stated by the students.

Rolle's Theorem

Rolle's theorem was introduced as an open-ended problem for exploration by the students. The problem was stated as follows: "Let f be a function defined on [a,b] with f(a) = f(b) = 0 . What conclusions can you draw about f?" The groups generated the following collection of theorems, which were not expected by the teacher.

1. f has either a positive maximum or a negative minimum, and the derivative of f is 0 at those points.

2. If f has both a positive maximum and a negative minimum value, then f takes on a zero value between the maximum and the minimum points. ("Between" refers to the x-axis.)

3. Between the positive maximum and the negative minimum, f has an odd number of zero values.

4. Between the positive maximum and the negative minimum, f has an inflection point. (Some of the students knew about inflection points from courses in high school.)

5. Between the positive maximum and the negative minimum, f has an odd number of inflection points.

With appropriate hypotheses, Rolle's theorem states that there is a point c in (a,b) such that f'(c) = 0 . Many students were disappointed to find this result labeled as Rolle's theorem. "You mean the least interesting theorem is the one with the name?"

All the theorems were false in the form initially stated by the students, since there were not enough hypotheses to justify the conclusions. The students initially stated no hypotheses concerning either the continuity or the differentiability of the function. In all the theorems the students implicitly made a continuity assumption to state the existence of maximum or minimum values. In (2) some students implicitly made a continuity assumption to use the intermediate value property. In (1) some students implicitly assumed the existence of the derivative. In (3) some students implicitly assumed that f or f' had finitely many zeros. In (4) and (5) some students made implicit assumptions about the existence and number of zeros of the second derivative.

In each case the teacher stated that the theorem was true, given appropriate hypotheses. In each case the students had difficulty in furnishing the proper hypotheses. Once again, it appeared that most of the students implicitly thought of a function as a continuous, differentiable function with only finitely many relative maxima and minima.

The Mean Value Theorem

The mean value theorem was introduced with the following question. "Let f be defined on [a,b] satisfying the hypotheses of Rolle's theorem, but remove the restriction that $f(a) = f(b) = 0$. What conclusions can you draw about f?" When the students had difficulty formulating a conclusion, the hint was given to draw a picture and to state the conclusion of Rolle's theorem in geometrical language. After stating that the tangent line at some point was parallel to the x-axis (in Rolle's theorem), a student in each group made the appropriate analogy and stated that the tangent line at some point was parallel to the chord line joining the endpoints of the curve.

After the groups expressed their conclusion as a mathematical statement involving a derivative, the hint was given to begin the proof by constructing a new function which satisfied the conditions of Rolle's theorem. Within a few minutes someone in each group set $h(x) = f(x) - g(x)$, where $y = g(x)$ was the equation of the line joining the endpoints $(a, f(a))$ and $(b, f(b))$. Until this time matters had progressed roughly in accordance with the expectations of the teacher. However, each group then made the following argument. By construction, $h(a) = h(b) = 0$. By Rolle's theorem there is a point c in (a,b) such that $h'(c) = 0$. Then $f'(c) = g'(c)$, and $g'(c)$ is just the slope of the chord line, namely $(f(b) - f(a))/(b - a)$. QED.

Much to the surprise of the teacher, the groups had completed the proof without any of the usual complications involving the equation of the line $y = g(x)$. At the time the teacher was overly concerned about mathematical rigor and he asked the students to justify their argument by finding the equation of the line. One group refused to do so on the grounds that it was unnecessary, but two groups complied with the request. Members of the two groups spent more than fifteen minutes trying to derive the complicated line equation and to take the derivative $g'(x)$ by means of that equation. In the process of differentiation, most of the students became very confused over the notation and forgot which symbols represented constants and which represented variables. In addition, when the teacher asked all the groups if the function h satisfied all the hypotheses of Rolle's theorem, the two groups using the line equation had much more difficulty than the other group in answering the question.

In summary, the student groups developed a proof of the mean value theorem which was much more clear and comprehensible than the proof found in many calculus texts. The teacher foolishly attempted to impose his own preconceived notions of mathematical rigor upon the students, contrary to the design in Chapter II. In the groups which accepted the imposition, clarity and conviction were replaced by confusion and uncertainty about the proof.

5.10 THE DISTINCTION BETWEEN THEORY AND PROBLEMS

Many students made a distinction between theory and problems, particularly during the first semester when there was a large amount of theoretical work. As one student put it, "Calculus should be 25 per cent theory and 75 per cent problems." As far as the teacher could determine, the distinction between theory and problems depended largely upon the presence or absence of arbitrary functions. For example, the students labeled the statement of Rolle's theorem as theory, although the teacher had not identified the statement as a theorem until after the students had proved it. However, the statement was made in terms of the arbitrary function f. On the other hand, the students were given the particular function $f(x) = \sin(x) + 1$ for $0 < x < \pi$, $f(0) = f(1) = 0$. The students were asked whether the function satisfied all the hypotheses and the conclusion of Rolle's theorem. Despite the reference in the question to the terms "hypotheses" and "conclusion," the students labeled the question as a problem, not as theory. Most of the students indicated their enjoyment of several problems of this type having specific functions, even though they did not especially like Rolle's theorem itself.

The student distinction between theory and problems was mentioned numerous times during the pilot study. Many students, unlike many professional mathematicians, expressed the belief that general propositions were worthwhile only if they led to the solution of particular problems with specific functions. By way of illustration, the theoretical development of the formula for the volume of a general surface of revolution did not appear to be of much interest in itself. However, most of the students went through the development with good will, stating that the theoretical development would permit them to calculate the volumes inside particular figures such as spheres and cones. To take a second example, the proof of the fundamental theorem of calculus was very hard for the students. Nevertheless, they went through a two-day struggle without making a single complaint such as "Why are we proving this theorem anyway?"

The teacher attributed the student willingness to persist in the difficult proof to the clear recognition that the use of the theorem would permit reasonable solutions to a wide variety of integration problems. Prior to the proof of the theorem, the students had unsuccessfully tried to evaluate $\int_0^{\pi} \sin x \, dx$ with the use of upper and lower sums, and they then acknowledged the need for a better method of evaluating integrals. Moreover, several students said that the theorem had to be proved, since the result was not at all obvious.

Although the students generally liked problems better than theory, they did not reject all aspects of the theory, as shown by the previous examples. Students sometimes were heard to make a distinction between useful theory and useless theory. "Some theory is okay, but some of it is useless." Useless theory was theory aimed at providing a rigorous foundation for topics which were already clear to the students. In addition to the definition of the limit, many students perceived as useless the development of the natural logarithm and the exponential function, starting with the definition $\ln x = \int_1^{x} \frac{1}{t} \, dt$. This beautiful development, when judged from the student viewpoint, offered no new information about the natural logarithm and forced people to think about familiar facts in a rather remote and difficult context. Rather than providing a new feeling of theoretical security about a previously elusive topic, the development appeared to undermine the students' existing confidence about logarithms. "I understood the logarithm pretty well until we did it with that integral; now I'm totally confused."

In conclusion, most of the students usually preferred problems rather than theory. They accepted as useful those general propositions which had applications to interesting problems with specific functions. They frequently disliked the difficult proofs of general propositions intended to "prove the obvious" or "straighten out things we already know."

5.11 SUMMARY

In summary, the present chapter has included a rather startling description of the difficulties encountered in a discovery approach to calculus by students with a history of better-than-average achievement in mathematics. Those difficulties, it is hoped, will not obscure the fact that the students, working together in small groups with limited guidance from the teacher, succeeded in proving the major theorems of the calculus and in developing the techniques for solving all the pertinent classes of problems. Although the students encountered some "unthinkable" difficulties, they also stated some insightful conjectures and came up with some problem solutions and proofs that were not previously known to the teacher. Moreover, in view of the frequently-mentioned student distinction between theory and problems, it appeared that most of the students for most of the time wanted to do calculus, rather than analysis. In the chapter which follows, the achievement and attitudes of the students are discussed in further detail.

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Chapter VI

DATA FOR THE EVALUATION OF THE PILOT STUDY

6.1 OUTLINE OF CHAPTER VI

The present chapter is divided into three parts. The first part includes the results of a one-hour final examination comparing the small group-discovery method with the lecture-discussion system. The second part of the chapter includes the results of the seven take-home examinations which were administered to the small group-discovery class. The third part of the chapter is an analysis of student responses to an open-ended questionnaire administered in the small group-discovery class.

6.2 COMPARISON BETWEEN TWO METHODS OF INSTRUCTION

Pertinent Information for the Comparison

As part of the evaluation for the pilot study, a final examination was given to students in the small group-discovery class and in another class. In the discussion which follows, the small group-discovery class is referred to as the experimental group. The control group consisted of 51 students who learned calculus via the lecture-discussion system. The 51 students were a subset of a lecture class which met with 1 professor for 3 lectures per week; on the other 2 days, the 51 students met in 4 separate discussion sections led by 4 different teaching assistants.

There were certain differences between the 2 groups with respect to the group composition and conditions pertaining to the examination. The 12 members of the experimental group were all volunteers for a special class; all of these students had grades of A or B in their high school mathematics classes. The students in the control group were not volunteers and had not been subject to any special entrance requirements for the class. In comparing the 2 groups, there was no attempt to control such variables as SAT scores, IQ scores, sex, or grades in high school mathematics.

The final examination was administered at the end of the second semester of the one-year pilot study. During that year, the students in the experimental group had taken no examinations or quizzes in class. Students in the control group had taken a final examination during the first semester, several hourly examinations during both semesters, and quizzes at the discretion of the various teaching assistants.

The experimental group took the final examination designed for the control group but not vice-versa. The examination involved the recall of certain facts and standard computational skills; it did not require the solution of any difficult problems or the formulation or proof of any theorems. The examination did not include material such as limits of sequences which was covered in the experimental group but not in the control group.* However, the 25 items on the examination did include 7 items covered in the control group but not in the experimental group; the members of the experimental group were told to skip these items. Thus, the members of the experimental group had somewhat more time available during the 1-hour test. The comparison was made on the basis of the 18 items common to both groups, and the examination was scored by the author. The set of questions on the examination can be found in Appendix A, with the 7 previously mentioned items marked by an asterisk.

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*In Chapter V, data has been presented pertaining to the topics covered in the experimental group and the topics covered in the control group. Conclusions about the amount of material covered in the two groups are presented in Chapter VII.

Results of the Comparative Final Examination

The raw scores, mean, median, and standard deviation on the final examination are presented in Table 6.1 . The data are presented for the experimental group, for the total control group, and for 4 subsections of the control group which were taught by different teaching assistants.

The mean and median for the experimental group are 55.25 and 55.5 respectively. The mean and median for the control group are 52.35 and 53.0 respectively. Hence, the mean and median are higher for the experimental group than for the control group. On a section-by-section basis, the mean and the median are higher for the experimental group than for each of the 4 sections in the control group, with the exception of one section in which the median is the same as that of the experimental group. In addition, the standard deviation is 7.59 for the experimental group and 11.57 for the control group. To further examine the difference in achievement between the experimental group and the control group, an F-test was run on the data, yielding F-ratio .679 with $p < .413$. Hence, the difference is not statistically significant.

TABLE 6.1

SCORES ON THE FINAL EXAMINATION

Group	Experimental	Control				Control $= \bigcup_{k=1}^{4}$ Control k
		1	2	3	4	
N = Number of Students	12	12	16	12	11	51
Raw Scores	68	67	72	67	67	
	66	65	69	65	61	
	60	60	66	64	59	
	59	58	66	64	57	
	57	57	65	53	53	
	56	57	65	53	51	
	55	54	55	50	46	
	53	54	52	47	45	
	52	51	49	44	40	
	49	49	49	43	37	
	46	45	48	34	27	
	42	45	48	17		
			46			
			44			
			38			
			32			
Total of Scores	663	662	864	601	543	2670
Arithmetic Mean	55.25	55.17	54.00	50.08	49.36	52.35
Median	55.5	55.5	50.5	51.5	51.0	53.0
Standard Deviation	7.59	7.00	11.86	14.66	11.77	11.57

Table 6.2 includes the number of perfect solutions achieved in each group for each item of the test, excluding those items which were omitted in the experimental group. From these data was computed the average number of perfect solutions per student in each group. The average number of perfect solutions per student in the experimental group is 12.50 . The corresponding numbers for the 4 control sections and the total control group are 12.25, 12.38, 10.92, 10.91, and 11.69 respectively. Hence, the average number of perfect solutions per student is higher in the experimental group than in each control section taken separately and in the total control group.

For each of the fractions 7/12, 8/12, 9/12, 10/12, and 11/12, the following question was asked: How many problems were solved perfectly by that fraction of the group? The data were obtained from Table 6.2 and are summarized in Table 6.3 (see footnote to Table 6.3). There was 1 problem solved perfectly by 11/12 of the control group, but there were 4 problems solved perfectly by 11/12 of the experimental group. There were 3 problems solved perfectly by 10/12 of the control group, but there were 6 problems solved perfectly by 10/12 of the experimental group.

TABLE 6.2

NUMBER OF PERFECT SOLUTIONS PER PROBLEM FOR EACH GROUP

Group	Experimental	Control				Control
		1	2	3	4	
Number of Students	12	12	16	12	11	51
Problem						
4	9	8	9	5	3	25
5	7	3	11	8	8	30
6	11	9	16	10	11	46
7	11	9	10	10	9	38
8	9	9	10	7	7	33
9	5	5	10	3	6	24
10	9	9	9	9	7	34
11	11	11	13	10	10	44
12	8	11	12	6	6	35
14	9	10	13	11	11	45
15	8	10	10	5	3	28
16	3	4	7	5	4	20
17	7	9	10	7	4	30
18	11	7	16	7	8	38
19	4	9	10	7	8	34
20	10	5	10	9	5	29
21	10	9	10	7	2	28
24	8	10	12	5	8	35
Total	150	147	198	131	120	596
Average Number of Perfect Solutions Per Student	12.50	12.25	12.38	10.92	10.91	11.69

TABLE 6.3

NUMBER OF PROBLEMS SOLVED PERFECTLY BY VARIOUS FRACTIONS OF THE GROUP

Fraction of Group	Number of Problems	
	Experimental	Control
$\frac{7}{12}$	15	13*
$\frac{8}{12}$	13	9
$\frac{9}{12}$	10	5*
$\frac{10}{12}$	6	3
$\frac{11}{12}$	4	1*

*In the computations for Table 6.3, the various fractions of 51--29-3/4, 34, 38-1/4, 42-1/2, 46-3/4--were rounded to 29, 34, 38, 42, 46 . For the 3 starred entries in the table, the rounding down served to increase the totals for the control group.

For each fraction the number of problems solved perfectly by that fraction of the group was higher for the experimental group than for the control group.

The performance of the experimental group was satisfactory on all problems except numbers 9, 16, and 19, which were solved perfectly by only 5, 3, and 4 students, respectively. Problem 9 was an integration problem which the students should have been able to do. The poor showing is accounted for by the fact that 4 students made an error with a minus sign but otherwise solved the problem correctly. Problems 16 and 19 had to do with parametric equations and polar coordinates, respectively. As mentioned in Chapter V, these topics were dealt with rather hurriedly in the experimental group.

In summary, on a comparative examination dealing with basic facts and computational skills, the students in the experimental group performed slightly better, but not significantly better, than the students in the control group. The average number of perfect solutions was higher for students in the experimental group than for students in the control group. However, there were certain differences between the two groups with respect to the group composition and the conditions pertaining to the examination. An interpretation of the results is presented in Chapter VII. There now follows a discussion of the take-home examinations given to the experimental group but not the control group.

6.3 RESULTS OF THE TAKE-HOME EXAMINATIONS

There were seven take-home examinations during the pilot study--three during the first semester and four during the second semester. The topics covered by the examinations were the following.

1. Limits of sequences
2. Limits and derivatives of functions
3. Theory of curves and curve sketching
4. Applications of the derivative; the differential, implicit differentiation, the definition of the limit
5. Logarithmic differentiation; certain integrals involving the logarithm, exponential, and inverse trig functions
6. Integration techniques
7. Analytic geometry

The examinations can be found in Appendix B. The first examination was the longest, having 16 items; the last examination was the shortest, having only 4 items. Each item on each examination was scored on a 10-point scale, with partial credit given at the discretion of the teacher. The raw scores for each examination are presented in Table 6.4 .

TABLE 6.4

RAW SCORES ON THE TAKE-HOME EXAMINATIONS

Exam Number	1	2	3	4	5	6	7
Possible Score	160	140	120	60	60	80	40
Raw Scores	134	136	99	55	59	80	40
	132	134	99	54	59	74	40
	128	133	95	53	55	74	40
	121	130	89	53	54	73	40
	121	129	88	52	54	73	38
	119	125	88	49	53	73	37
	118	118	85	49	51	73	35
	113	117	84	48	50	72	35
	104	115	80	45	49	69	35
	103	115	80	45	48	66	31
	102	111	80	39	48	65	31
	96	99	77	39	44	61	31

The scores converted to a standard scale of 100 points are presented in Table 6.5, together with the mean, median, and standard deviation for each examination.

TABLE 6.5

RESULTS OF THE TAKE-HOME EXAMINATIONS

Exam Number	1	2	3	4	5	6	7
Possible Score	100	100	100	100	100	100	100
Converted Scores	84	97	83	92	98	100	100
	83	96	83	90	98	93	100
	80	95	79	88	92	93	100
	76	93	74	88	90	91	100
	76	92	73	87	90	91	95
	74	89	73	82	88	91	93
	74	84	71	82	85	91	88
	71	84	70	80	83	90	88
	65	82	67	75	82	86	88
	64	82	67	75	80	83	78
	64	79	67	65	80	81	78
	60	71	64	65	73	76	78
Total Scores	871	1044	871	969	1039	1066	1086
Mean	72.6	87.0	72.6	80.8	86.6	88.8	90.5
Median	74.0	86.5	72.0	82.0	86.5	91.0	90.5
Standard Deviation	7.90	7.97	6.30	9.16	7.53	6.38	8.92

Discussion of the Results of the Take-Home Examination

Since the teacher neglected to collect data on individual items, the discussion must be restricted to the results taken as a whole. The standard deviations on the 7 examinations ranged from 6.30 to 9.16 . In 5 of the 7 examinations, the means and the medians were each higher than 80 on a standard scale of 100 points. The exceptions to this trend occurred on Examinations 1 and 3 . On Examination 1, the mean and the median were 72.6 and 74.0 respectively. On Examination 3, the mean and the median were 72.6 and 72.0 respectively.

Although item data was not collected, a glance at the measuring instruments is sufficient to explain the lower results on the 2 cited examinations. On Examination 1, item 2 is rather tricky and items 9, 11, 15, 16 are quite difficult. On Examination 3, item 4 is difficult and items 5 and 6 are quite difficult. All the other examinations include a smaller proportion of difficult items.

Table 6.5 includes 84 individual scores. Among the 84 scores, only 10 scores are lower than 70 . The absolute minimum score is 60, which occurs only once on Examination 1. Moreover, the records of the teacher show that no member of the class was the low scorer for more than 2 of the 7 examinations.

The interpretation of the results of the take-home examinations is made in Chapter VII. A discussion now follows of the student responses to an open-ended questionnaire.

6.4 THE QUESTIONNAIRE

A ninety-item open-ended questionnaire was used to determine the student reactions to the small group-discovery class. One purpose of the questionnaire was to determine whether or not the design in Chapters II and III was successfully implemented during the pilot study. Some of the important issues were the following. Did the students perceive every mathematical topic as being either interesting or useful? How effective was the teacher in implementing the democratic style of leadership? Was there much competition in the work groups? Were the work groups too small or too large or neither? Did the grading system in fact reduce anxiety? Other questions were asked about issues not specifically mentioned in the design phase. Did the method have any effect upon the student's interest in mathematics? Did problem solving every day become boring? How stimulating was the calculus class in comparison with other classes?

In constructing the questionnaire, it was decided that open-ended items might provide a good deal of information not readily available in another form, although it was recognized that such items would be difficult to classify and count. The questionnaire itself can be found in Appendix C. The questionnaire is a loosely constructed instrument and no claim is made about its reliability. Clearly, the instrument is not suitable for use in a carefully controlled investigation. Nevertheless, for an informal evaluation of the first trial of a new instructional method, it seems to furnish the necessary information.

The questionnaire is divided into five major sections, dealing respectively with the work groups, the mathematics content, the teacher, various practices and policies, and basic reactions to the class. The questionnaire was given to the students about two weeks before the end of the second semester and was to be returned to the author no later than the time of the final examination.

For almost every item on the questionnaire, the student responses were classified into various categories and counted, although for certain items it was not possible to form very neat categories. In almost all categories, at least one student response was selected for quotation in the paper; any response which was quoted was done so in its entirety. If there were several interesting responses within a category, the author selected responses representing the variations in student viewpoint within that category. Responses which were one or two words long or quite mundane were almost never selected for quotation. Examples of unquoted responses were, "yes," "no," "very much so," "I didn't like it."

The tabulated responses and the quoted responses are presented in Appendix D. The body of the paper includes an outline of the structure of the questionnaire and an item summary of the student responses. To convey the spirit of the student responses, quotations are presented in the text for a small selection of the items.

The Structure of the Questionnaire

1. The Work Groups (items 1-39)
    a. The initial acquaintance with others (items 1-3)
    b. Group size (4-5)
    c. Group composition and formation (6-12)
    d. Behaviors and moods of other group members (13-16)
    e. Effective group functioning (17-18)
    f. Communication (19-22)

   g. Involvement (23-24)
   h. Interpersonal comparisons (25-26)
   i. Cooperation versus competition (27-30)
   j. Leadership (31-35)
   k. Physical factors (36-38)

2. The Mathematics Content (items 39-43)

3. The Teacher (items 44-55)

   a. The teacher as work supervisor (44-49)
   b. The relationship between students and the teacher (50-55)

4. Various Practices and Policies in the Class (items 56-73)

   a. Problem solving versus lectures (56-57)
   b. The notes in place of a textbook (58-63)
   c. Homework (64-67)
   d. Exams and grades (68-73)

5. Basic Reactions to the Calculus Class (74-90)

   a. Skill in problem solving (74-76)
   b. Comparisons with other classes (77-82)
   c. Certain changes within the class (83-84)
   d. Mathematical goals and interpersonal goals (85-88)
   e. Weaknesses and strengths of the method of instruction (89-90)

Since the flavor of the student responses cannot be conveyed by the item summary, there now follows a brief section which includes some quoted responses to four questions. Each item is presented in exactly the form in which it appears in Appendix D.

Student Responses to Four Items

85. What effect if any has this method of learning had on your interest in mathematics?

Categorized Responses	Frequency
a. No change	6
b. Increased interest	4
c. Other comments	2

Sample Responses

a. None. I've always loved it and always will.

I have always been interested in math and I think this class has helped maintain my interest by giving me the opportunity to understand what was happening.

I think my attitude for math is the same as it was in high school. If I am able to understand it and what is going on I like and want to learn it, but if I have troubles with understanding then my interest declines.

b. It showed me that I can do things that before looked impossible. All it takes is a little understanding. Math doesn't scare me as much now.

Any calculus course which made me seriously consider changing my major to mathematics necessarily has had a great effect.

c. I must admit it made it more fun but I still wouldn't get turned on about it.

Sorry but none--has made teaching a more reasonable profession though.

82. How stimulating has this class been in comparison with other classes?

Categorized Responses	Frequency
a. The same	2
b. Better or more stimulating	5
c. Much more stimulating, great, or no comparison	5

Sample Responses

a. It has been very stimulating, but about the same as my other classes since I had very good classes.

b. This class has been more stimulating than other classes (especially lectures).

c. Much more stimulating, because of its unusual handling of the material.

It is my most interesting and liked class. I enjoy coming to it.

Forget it there's no comparison. It was great.

86. How did working with other students influence your learning?

Categorized Responses	Frequency
a. Uncertain	2
b. Positive Effects	10

Sample Responses

a. It is very difficult to judge.

b. Other students, no matter who, force you to learn more.

It helped me because I gained confidence showing people how to work a problem, also realized my limitations when people showed me how to do a problem.

A lot of times when I did not understand something the other members of the group helped to clear things up.

The working out of problems together not only removed much of the frustration of working difficult ones by oneself, but it also helped keep up a constant renewal of interest.

I learned to depend on working it out myself or with the help of others instead of relying on a book. In other words I think I developed a little original thinking.

I think I learned a lot more this year than I did in all three years of high school math.

I think you learn from students, while you're taught by teachers. I think you know what I mean. With a student you understand, with a teacher you too willingly accept.

81. Did you feel that the objectives of this class were different from other math classes? If so, in what way?

No. We were assured that we would complete the same curriculum.

The main objective for us was to learn calculus. A secondary objective for us but a primary objective for the teacher was to test the method of teaching calculus.

Yes. Instead of making students learn, it tried to instill into its members a desire for learning math.

Yes. It lowered grade consciousness and underscored theory.

Yes, the other classes were more grade oriented while this class was learning oriented.

The objectives of a class would be an understanding of calculus no matter how it was conducted. However, in this class there was a greater emphasis on the students learning rather than being taught.

Yes in that we could get more out of the course with solving problems together.

Yes the student was supposed to understand not memorize.

Yes. We were encouraged to understand better.

Yes. Most classes stress being able to use formulas while this stresses total understanding.

In this class we strived to understand the meanings of the theories, whereas in other math classes only a skill in manipulating formulas was required.

Only in that we were to enjoy it more while learning and were learning to work better with people while we learned the math. Also we concentrated more on the methods of math rather than memorizing specific formulas--a goal very different from other math courses and one I agree with completely.

6.5 SUMMARY OF RESULTS FROM THE QUESTIONNAIRE

THE WORK GROUPS

The Initial Acquaintance with Others

1. Everyone reported that the name tags helped in getting to know others at the beginning of the class. For ten people, the process took no more than two weeks. The maximum time mentioned was six weeks.

2. Four people would have liked some kind of extra session to get to know the others at the beginning of the year; eight people would not.

3. Three people knew others before the class began and reported a favorable effect of the previous acquaintance. Eight others were unaffected, but one person reported being bothered that others already knew each other.

Group Size

4. Five people reported little or no effect when one member was absent from their group. Five others reported some detrimental effect, and two said the effect depended on the particular member who was absent.

5. Six people stated a preference for the four-member work group, and five people were willing to consider the three-member group as a reasonable possibility. Everyone who responded directly to the question rejected the groups with two or five members as being respectively too small and too large.

Group Composition and Formation

6. Six people chose their group members on the basis of personality and social compatibility; three others considered these factors plus ability. One person considered only the rate of working problems, and two didn't care whom they worked with.

7. Seven people preferred coeducational groups, two preferred groups with all boys, and three had no preference.

8. A desire to avoid working with certain class members was expressed by nine people. The undesirable characteristics that were mentioned included being obstinate, refusing to share ideas, being headstrong, refusing to listen to criticism, not providing explanations, skipping steps, being lazy, working at a different pace, consistently using different approaches in problem solving, and various personality clashes.

9. Five people said that they were not very often in a group with someone they didn't like. Two people said they were often in such a group, and three said they were never in such a group. Only one person reported undisturbed functioning when in a group with a disliked member. Others mentioned effects such as loss of interest, tension, frustration, and mental and emotional withdrawal from the situation.

10. The schedule of changing groups roughly every three weeks was satisfactory, according to eight students. The others said that groups should change when the topics change or when individuals want a change.

11. Seven people stated that individuals should be able to change groups without having everybody change; with certain conditions, three other members agreed with the statement.

12. Eleven people did not like the suggestion of picking a desirable group and working in it all the time. Their comments included such matters as clique formation, boredom, intellectual stagnation, laziness, and, also, the positive effects of switching groups--fresh approaches to problems, learning to work with all kinds of people, and the reduction of competition between groups.

Behaviors and Moods of other Group Members

13. Three group members reported no displeasing behavior by others. The other nine members each mentioned some of the following behaviors: being too quiet, being lazy,

not doing homework, cutting class, goofing off, acting childish, being slow at simple arithmetic, hogging the chalk, not listening to others, acting domineering, acting superior, not explaining ideas upon request.

14. When other people's behavior was bothersome, eight of the students did nothing, ignored it, or tolerated it. Two people gave very mild hints, and two made direct complaints.

15. Bothersome behaviors of others had only a slight effect upon three members, but a definite detrimental effect upon six members. The effects included reduced working efficiency, lowered interest, withdrawal, tendencies to fight back, and reduced group unity.

16. Seven people reported being affected by the moods or attitudes of others. The responses included concern over dejected moods, a subdued group atmosphere, and a dislike for superior attitudes.

Effective Group Functioning

17. Four people very cautiously stated that more time might have been spent to improve the group functioning, particularly at the beginning of the year. The other eight people believed that nothing more could or should have been done; several of them said that working in a group can be learned by experience.

18. Effective group functioning was promoted by cooperation and by the presence of one group member who was enthusiastic about mathematics; these factors were each mentioned by three students. Other factors were attentiveness, being prepared, common interest, joint problem solving, contributions from everyone, a sense of humor, the absence of ridicule and of rebukes, lack of violent personality clashes, and sufficient familiarity to express one's feelings.

Communication

19. Eight people reported feeling completely free to ask questions when they didn't understand something. Four

people felt very free, but this depended on the person being asked.

20. Seven people stated that others usually provided satisfactory explanations. Four people said it depended on the particular explainer, and one said that the explanations were usually not adequate.

21. If they didn't understand something, five people preferred to ask another group member, rather than the teacher. Four people would ask a group member and then the teacher, if necessary; three preferred to ask the teacher.

22. People usually listened to one another, according to nine students. Three students stated that listening sometimes occurred, depending on certain conditions in the group.

Involvement

23. Nine people reported an adequate opportunity to express their ideas in their groups. Three people said it depended upon the group.

24. Five people said that they never felt left out of their group, but seven people reported feeling left out sometimes--if everyone else understood, if their question was left unanswered, if they were unprepared, or if they didn't feel interested in the mathematics.

Interpersonal Comparisons

25. Six people were aware of the abilities of others relative to themselves, and three people were very much aware of this comparison. Only one person said he was not concerned about it.

26. Three people were bothered if other members understood things more quickly, four were not bothered, and five were not bothered except in certain conditions.

Cooperation versus Competition

27. Five people reported competing with others, four said they did not compete, and three said they did not compete except under certain special circumstances.

28. Four people said that there was much competition in their groups and that competition had a stimulating effect. Six people said there was not much competition.

29. Eight people were aware of the progress of other groups relative to their own. Six people said it mattered; five said it didn't.

30. Seven people reported that their group competed with other groups, and three people said their group did not compete. No one mentioned any bad effects of intergroup competition, except for frustration when they were behind.

Leadership

31. Six people felt involved in the problem when somebody else was writing the solution. Four people said they felt involved under certain conditions--if they understood, if they contributed something, or if the writer stayed in touch with the rest of the group. Three people expressed a preference for writing themselves and two people preferred to watch.

32. The group functioned more effectively with everybody at the blackboard, according to five students, and with one person at the blackboard, according to four students. The others said it depended on the group and/or the problem.

33. Eleven students said there were certain groups where one person was clearly the leader. Three students preferred or tolerated having a leader, six students disliked having a leader, and two said it depended on the particular leader.

34. Six students disliked the practice of having experts on particular problems, three advocated this practice, and the others did not give clear-cut responses.

35. Two people liked being an expert, three liked it with conditions, four disliked it, and the others were non-committal. While some people got satisfaction from completely understanding and presenting a solution, others felt that they were in the spotlight, were nervous, neglected the other problems, or preferred a solution by the whole group.

Physical Factors

36. Six people were for and six people against the idea of everyone having his own piece of chalk. The proponents said this practice would promote the contribution of new ideas and the making of corrections without having to take the chalk away from someone. Others believed that too much chalk would lead to chaos, lack of cooperation, and disintegration of the group.

37. Six people had definite preferences for the location of their group in the room, and six others said it didn't matter. Some people liked the corner or the side board, while others preferred the middle of the front board.

38. If they sat down, six people preferred to be within five feet of the blackboard. Two people preferred to be more than six feet away. Four people specified no particular distance, but wanted to be close enough for various purposes--seeing, hearing, or participating.

REACTIONS TO THE MATHEMATICS CONTENT

39. There was no clear preference for solving either many short problems or a few long problems during the class period. Three people liked short problems, four liked long problems, three liked a mixture, and two said it depended on the particular problem.

40. Six people did not enjoy the theory (for example, formulation of definitions and proofs of theorems). Four people enjoyed the theory, and two others enjoyed it only sometimes.

41. Six people preferred computational problems rather than proofs, two people preferred proofs, and two preferred a proof followed by computations. Eleven people reported no change in their preference during the year.

42. Three people were never concerned about covering enough material during the year. Nine people were concerned for at least one semester, and five of these for the entire year.

*43. The students indicated the following preferences for nine major topics of calculus (arranged in decreasing order of preference):
(1) computation of derivatives, (2) integration techniques, (3) analytic geometry, (4) vectors, curve sketching, theorems about curves, (5) definition of a limit, logs and exponentials, (6) inverse trig functions.

Only computation of derivatives and integration techniques were strongly liked by the students. Analytic geometry was liked by more than half the respondents. Vectors, curve sketching, and theorems about curves were in the neutral range. The definition of a limit, logs and exponentials, and inverse trig functions were in the dislike-to-neutral category.

The preferences were determined from the responses to item 43 in Appendix D. Weights of 4, 3, 2, 1 were assigned respectively to each category, with "dislike" receiving a weight of 1 and "strongly like" receiving a weight of 4 . For each topic a weighted sum was then computed. The sums appear in the following table.

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*The data are based on responses from 10 out of 12 students. The responses of 2 students were very incomplete and could not be categorized.

TABLE 6.6

STUDENT PREFERENCES FOR TOPICS IN CALCULUS

	Topic	Weighted Sum
1.	computation of derivatives	37
2.	integration techniques	36
3.	analytic geometry	27
4.	vectors curve sketching theorems about curves	22
5.	definition of a limit logs and exponentials	19
6.	inverse trig functions	17

Maximum possible score: 40 Minimum possible score: 10

STUDENT REACTIONS TO THE TEACHER

The Teacher as Work Supervisor

44. Ten students replied that the teacher spent the right amount of time with each work group. One student would have preferred more help at times.

45. Everyone liked to have the teacher check the group problem solutions. Four people always waited to have the solution checked before doing another problem; the others sometimes waited.

46. Seven people said the teacher was effective in giving hints, and five people said he was sometimes effective. People in the latter category mentioned that hints could be worded differently or be more apparently related to the problem. No one wanted more hints.

47. Seven people reported no need for the teacher to exert more influence over behavior. Five people felt that more influence was necessary by the teacher--quieting people when he talked, being more strict about homework, suggesting improvements for group functioning, and speaking to particular individuals when necessary.

48. The teacher helped the group to solve problems by giving hints, according to eleven respondents. Other helpful behaviors were: pointing out errors, not allowing students to give up, providing encouragement, "discouraging a few times when it was blatantly needed," and not taking over the group.

49. The teacher did nothing to interfere with progress in the work groups, according to eight people. Four people reported some interference: "sometimes stressing seemingly trivial parts of proofs," "sometimes stifled new ideas for the sake of the conventional method," sometimes being with one group when help was needed in another, and sometimes observing too closely.

The Relationship between Students and the Teacher

50. Seven students developed trust in the teacher within a few weeks or within some short time. The other students took longer or mentioned no specific time. Trust was facilitated by: the first-name basis, equalitarian behavior, expressions of confidence in the students, honesty, student freedom to ask questions, group discussions, and talking of things other than math.

51. Nine students found it hard or very hard initially to call the teacher by his first name. Ten students thought that the first-name basis had a beneficial effect, but one thought it involved lack of respect for the teacher.

52. Three people reported feeling free to call up the teacher about problems. Six others did not call him.

53. Everyone perceived the teacher more as a helper and guide than as an evaluator and critic.

54. Nine students had no criticisms that they did not express to the teacher. Three students expressed their criticisms on the questionnaire: the notes should have been typed, the teacher should have controlled the behavior of certain individuals, and he should have pushed harder on homework.

55. Eleven students reported a closer, more personal relationship with their math teacher than with other teachers.

VARIOUS PRACTICES AND POLICIES IN THE CLASS

Problem Solving versus Lectures

56. Ten people said that working problems every day did not become routine and monotonous. For the other two people, problem solving sometimes became monotonous but relations with others did not.

57. Five people saw no need for any more lectures. Four people preferred more lectures, at least occasionally. The lectures could have served the following purposes: to give a start on problems, to summarize a topic, to break the routine, and to cover more material. Three people did not directly answer the question.

The Notes in Place of a Textbook

58. All students found the notes to be either useful or very useful. Three students found defects in the notes, namely a lack of detail, problems with the author's handwriting, and the lack of an index.

59. Only half of the group read the notes as often as once per week. The other students read the notes only when confused or uncertain, after missing a day, or before the review sheets.

60. Nine students either would have or might have used the notes differently if they had been available on the same day. This would have been done to resolve points of difficulty and to assist in solving homework problems.

61. Five people took a few notes in class, but no one took many notes. The people who took a few notes did so in order to recall points of difficulty and certain formulas or processes needed for the homework. Several people pointed out that note-taking aids recall but hinders either speed or willingness to experiment.

62. Seven people were not bothered by the lack of a textbook, while the other five were definitely or somewhat bothered. The latter set mentioned their previous reliance on learning from books and the need for an index.

63. Three people read a calculus book at some time during the year, two of them with some regularity. The reading was done to solve homework problems when notes were not available and to provide another introduction to new topics.

Homework

64. Half of the group members were usually able to work homework problems dealing with new topics, three were sometimes able, and three encountered frequent difficulties. The people with difficulties reported the need for a slower introduction, a simple example, or practice in class before doing the homework.

65. Five people reported doing at least four hours of homework per week, four did less, and the others did whatever was necessary; one person did all the problems, regardless of time. No one felt overworked, five people felt they did the right amount of homework, and five others felt that more was needed, particularly in the first semester. Several people believed that little or no homework was necessary, since the problems were solved in class.

66. Six people felt they were not doing the right amount of homework for various reasons, including laziness, involvements elsewhere, and lack of necessity to do more.

67. Everyone discussed calculus problems with others at some time outside of class--seven in person and five by telephone. Two people did this regularly, two whenever necessary, and the rest less often.

Exams and Grades

68. Six people said review sheets should have been given every 2 weeks or every 2-3 weeks. Each of the following responses was made by two people: more often, the timing was satisfactory, after each topic with no fixed time schedule.

69. One person was not concerned with the performance of others on the review sheets, seven people were slightly concerned, and four others were rather or very concerned.

70. One person said that there should have been exams or quizzes in class, eight people said there should not, and three were uncertain. Those in favor of exams or quizzes mentioned the importance of committing things to memory. Those opposed mentioned fierce competition, over-consciousness with grades, time-wasting with exams, thinking more clearly without pressure, learning from the review sheets, and exams making this class into "another uninteresting class."

71. Three people reported feeling no grading pressure at all, four felt not much, two felt much less than in other classes, and one person felt pretty much pressure. Four people said that no grading pressure is desirable, and five people said that some pressure is necessary.

72. Seven people were either generally unconcerned with grades or less concerned than in their other classes. Three people showed substantial concern about their calculus grade.

73. Six people stated that a guaranteed A grade would not have influenced the amount of work they did in calculus. Two people were not certain of the effect, and four mentioned a probable or definite decrease in their level of work.

BASIC REACTIONS TO THE CALCULUS CLASS

Skill in Problem Solving

74. Seven people felt that there was an increase in their problem solving skill during the year, one felt there was a probable increase, and one was uncertain. Three people reported no increase but mentioned other benefits--faster development and understanding of new ideas, increased confidence and desire to solve problems, and increased ability to think on her feet and experiment.

75. Eleven people said there was enough opportunity to solve problems independently; the other person blamed himself for not doing enough homework.

76. Four people said that solving problems with others had increased their ability or desire to solve problems by themselves; two said they now preferred working with others rather than working alone. Three people said there was no effect, one made a complete separation, and two were uncertain.

Comparisons with Other Classes

77. Seven people said their attendance was better in the math class than in other classes; five people said there was no difference.

78. Ten people reported no conflict between this class and their other classes; two people reported less satisfaction with certain aspects of their other classes.

79. Five people reported negative reactions by students in other classes upon hearing about this class. One person reported surprise, four people reported mixed reactions, and two reported positive reactions.

80. Six people saw other class members socially, two people seldom did so, and four did not. Three people made more friends in this class than in other classes, two people made slightly more friends in this class, two

people said there was no difference, and five made no clear comparison.

81. Eleven people felt that the objectives of this class were different from other math classes. The contrasts included the following: instilling a desire to learn, emphasizing learning rather than grades, learning rather than being taught, understanding rather than memorizing or manipulating formulas, enjoying the learning process, learning to work with people, and testing the method of instruction.

82. Five people said this class was better or more stimulating than their other classes, five people said it was much more stimulating, and two others said there was no difference.

Certain Changes within the Class

83. Five people reported no change in their attitude toward the class during the year, and seven people reported a change for the better. The latter group mentioned increased appreciation and decreased worrying as the members became better acquainted with each other, with the teacher, and with the method.

84. Eight people mentioned that the class behavior changed during the year in the direction of a more relaxed atmosphere with better group functioning. Four people mentioned other changes--more awareness of the concerns of others, more seriousness about goals, and more enthusiasm for problem solving.

Mathematical Goals and Interpersonal Goals

85. Six people said the method had no effect on their interest in mathematics, and four people said their interest increased. Two people mentioned other changes: teaching became "a more reasonable profession," and math was more fun, within limits.

86. Two students were uncertain, but ten students said that working with others influenced their learning in a positive way.

87. The opportunity to interact with class members had little value for one person, moderate value for five people, and great value for six people.

88. Seven people said that the mathematics was more important than their relations with other group members, two said that both were important, and three people said that the human relations aspect was more important.

Weaknesses and Strengths of the Method of Instruction

89. Seven students said they would have no reservations about taking future courses taught by this method. Five people expressed reservations, which were the following: doubt about learning as much as in ordinary classes, the great dependence of this method upon the particular instructor, the desire to try a lecture-quiz course in math, the wish to avoid contact with other people, and fear of not being able to make it in a regular class.

90. The students mentioned the following advantages of the method: less grade pressure, more interesting, easier, opportunity to clear up questions, more fun, challenging, builds good relationships with others, greater student-teacher contact, stimulates desire for math, gain ideas from other people, learn to teach, get more help, learn more thoroughly, develop greater understanding, think for oneself, opportunity for creative thinking.

6.6 SUMMARY OF IMPORTANT RESULTS

Mathematical Achievement

During the class meetings, the students in their work groups proved the theorems of calculus and developed the techniques for solving various classes of problems.

In a comparative final examination dealing with basic facts and computational skills, the students in the small group-discovery class scored slightly better, but not significantly better, than the students in a control section taught by the lecture-discussion system.

On 5 out of 7 take-home examinations in the small group-discovery class, the means and the medians were each higher than 80 on a standard scale of 100 points. On the remaining 2 examinations, which were quite difficult, the means and the medians were each higher than 70.

Affective Responses Pertaining to the Learning of Mathematics

Two thirds of the class members either did not enjoy the theory or enjoyed it only sometimes (Q 40).

No student reported a decrease in his interest in mathematics during the pilot study and one third of the students reported an increase in their interest in mathematics (Q 85).

No student reported a decrease in his skill in problem solving during the pilot study and more than half of the class members felt that there was an increase in their problem solving skill (Q 74).

Most of the students said that working with others had positive effects upon their own learning (Q 86).

Only one fourth of the class members were never concerned about covering enough material during the pilot study (Q 42).

Student Reactions to the Teacher

Most of the students reported that the teacher spent the right amount of time with each work group and gave enough hints (Q 44, Q 46).

More than half of the students said that the teacher was effective in giving hints, and the other students said he was sometimes effective (Q 46).

Everyone perceived the teacher more as a helper and guide than as an evaluator and critic (Q 53).

Almost all the students reported a closer, more personal relationship with their mathematics teacher than with their other teachers (Q 55).

The Work Groups

All the students believed that the two-member group was too small and the five-member group was too big. There was a division of opinion about the relative merits of the three-person group versus the four-person group (Q 5).

Three fourths of the class members expressed a desire to avoid working with various individuals (Q 8).

More than half of the class members were sometimes in a group with a person they didn't like (Q 9).

Two thirds of the class members reported feeling completely free to ask questions when they didn't understand something. The other people felt very free, but this depended on the person being asked (Q 19).

Three fourths of the class members said that they had an adequate opportunity to express their ideas in their groups. The other people said it depended upon the particular group (Q 23).

Almost half of the class members reported competing with others (Q 27, Q 28).

While some of the groups functioned very well, others did not (Q 8, 9, 13-16, 19, 23, 33).

Various Practices and Policies in the Class

Most of the class members said that working problems every day did not become routine or monotonous (Q 56).

Three fourths of the students said that they never read a calculus book during the pilot study (Q 63).

Three fourths of the students reported feeling not much grading pressure or much less pressure than in their other classes (Q 71, Q 72).

Basic Reactions to the Calculus Class

Half of the class members saw other class members socially (Q 80).

Most of the students said that their calculus class was better, more stimulating, or much more stimulating than their other classes (Q 82).

More than half of the class members said that their attitude toward the class changed for the better during the year, and the other students said there was no change in their attitude (Q 83).

Chapter VII

CONCLUSIONS AND PROBLEMS FOR FUTURE INVESTIGATION

7.1 OUTLINE OF CHAPTER VII

In Chapter VI some data on student achievement and attitudes have been presented for the evaluation of the pilot study. The present chapter includes a report of conclusions which follow from the data. A number of problems are suggested for future investigation with the small group-discovery method, and the chapter ends with a summary of the paper.

7.2 CONCLUSIONS

Results of the Comparative Final Examination

On a comparative final examination dealing with basic facts and computational skills, the members of the small group-discovery class performed slightly better, but not significantly better, than the students in a control section taught by the lecture-discussion system. There are several factors, however, which make it rather difficult to interpret this result.

1. The students in the small group-discovery class were all volunteers for a special section, and they knew this special section was an experimental class. Therefore, the performance of the discovery class could be attributed to the Hawthorne effect.

2. The students in the discovery class were at least mildly interested in mathematics and had grades of A or B in all their high school mathematics courses. The students in the lecture class did not have to meet these special entrance requirements. Hence, the comparative performance of the two groups could be attributed simply to the differences in their composition.

3. Members of the small group-discovery class were told to omit certain items on the examination. The members of the discovery class therefore had somewhat more time during the test than the members of the lecture class, and the comparative performance could be attributed to the time difference.

4. To balance the picture somewhat, the students in the discovery class had taken no quizzes or examinations in

class during the entire one-year pilot study. Students in the control section had taken a final examination during the first semester, several hourly examinations during both semesters, and quizzes at the discretion of the various teaching assistants. Thus, there was a practice effect operating in favor of the lecture class.

After stating all these qualifications, it is time to come to the heart of the matter. The one year pilot study was a first step in determining the feasibility of a new instructional method. A comparison was made between the small group-discovery method and the lecture-discussion system to see if there was any loss in student achievement under the new approach. In terms of this comparison, there is no cause for discouragement about the effects of the new instructional method upon student achievement. It seems entirely appropriate to proceed with further investigations.

The Rate of Coverage of Material

In Chapter V an outline was presented of the material covered in the small group-discovery class. There was also a listing of the topics covered in one large section of a course taught by the lecture-discussion system but not covered in the small group-discovery class. If one bears in mind the fact that the small group-discovery class had a one-week review and a five-week unit of "extra" material on limits of sequences, the following conclusion emerges: the students in the small group-discovery class did not learn less subject matter than the students in the lecture class. Since everyone "knows" that the discovery approach is slower than the lecture approach, some explanations are in order. The explanation is probably a combination of the following set of factors.

1. As previously mentioned, students in the discovery class had to satisfy certain entrance requirements not imposed upon students in the lecture class.

2. As previously mentioned, there may have been a Hawthorne effect in operation.

3. Some of the students in the lecture-discussion system took quizzes during their discussion sections. Moreover, the lecture class had several examinations, each of which entailed two-to-three days of class time for review, testing, and presentation of the solutions. The students in the discovery section had no quizzes or examinations in class and spent very little time discussing the results of the take-home examinations.

4. The teacher of the lecture class may have presented some material of cultural value which he wanted the students to see but not to master. No such material was presented in the discovery class, since the students did the mathematics in their groups.

5. The additional material on sequences in the discovery class may not have all been "extra" material. It appeared that there was a considerable transfer of learning from the material on sequences to the material on functions of a continuous variable. In dealing with limits of sequences, the students learned certain computational skills and limit theorems and spent considerable time in discussions of the limit concept. The comparable material for limits of functions was then handled rather quickly. Moreover, the unit on sequences laid the foundations for the definite integral and for the informal development of the derivative of the logarithm function, using properties of the number "e." Hence, the time initially devoted to sequences appeared to speed up the coverage of some of the later material.

6. Conceivably, the small group-discovery approach could permit more rapid coverage of material than an individual discovery approach.

After stating all these explanations and qualifications, it seems fair to conclude that the students in the small group-discovery class did not learn less material than the students in the lecture class. However, the students in the discovery class were probably exposed to less material than the students in the lecture class. If one places emphasis on the learning of subject matter, in contrast to exposure to subject matter, it again seems appropriate to proceed with further investigations of the small group-discovery method.

Results of the Take-Home Examinations

There might be an objection to drawing conclusions about student learning on the basis of the comparative final examination. That examination was only one hour long, it was rather simple, and it entailed only the recall of certain basic facts and computational skills. In addition to the final examination, the students in the small group-discovery class took seven take-home examinations. The set of questions for these examinations can be found in Appendix B. In the opinion of the author, these instruments are non-trivial tests for beginning students of calculus. On the basis of that observation and the achievement data presented in Chapter VI, the conclusion follows that the students learned some mathematics during the pilot study.

Conclusions Based on the Questionnaire

The teacher intended to provide a learning environment in which the students perceived every mathematical topic as being either interesting or useful. This goal was definitely not achieved.

Most of the class members were concerned about covering enough material during the pilot study.

The small group-discovery approach had either non-negative or positive effects upon each student's interest in mathematics.

The small group-discovery approach had either non-negative or positive effects upon each student's perception of his problem solving skill.

Most students believed that working with others had positive effects upon their own learning.

The attempt to apply the democratic style of leadership in the mathematics class was basically successful.

Almost all the students had a closer, more personal relationship with their mathematics teacher than with their other teachers. In this respect the small group approach functioned as a counterforce to anonymity, loneliness, and alienation.

The attempt to promote cooperation was only partially successful.

While some of the groups functioned very well, others did not. The formation of effective and satisfying working combinations was one of the major difficulties encountered during the pilot study.

Most of the class members felt free to ask questions when they didn't understand something and also felt that they had an adequate opportunity to express their ideas in their groups. Hence, the functioning of the work groups was basically successful in these respects.

For most of the class members, working problems every day did not become routine or monotonous.

Most of the students did not feel much grading pressure or felt much less pressure than in their other classes. Hence, the attempt to reduce anxiety via the grading system was largely successful.

Half of the class members saw other class members socially. In this respect the small group approach was partially successful as a counterforce to anonymity and loneliness, not only during the class meetings but also outside of class.

For most of the students, their calculus class was better, more stimulating, or much more stimulating than their other classes.

The attitudes of students toward the class either remained the same or changed for the better during the year. Thus, the appeal of the approach did not wear thin over time.

A further indication of the success of the small group approach was that a number of students asked for an extension of the class for another semester. The Department of Mathematics kindly consented to schedule a third semester special course for the members of the small group-discovery class.

On the basis of the evidence, it seems fair to conclude that the pilot study was a successful first attempt to implement a new method of mathematics instruction. There were numerous indications of the desirability of proceeding with further investigation of the small group-discovery method. Accordingly, it is appropriate to present a number of questions for future investigation concerning the method. Some of the questions deal with those aspects of the method which were least successful during the pilot study, and other questions pertain to those aspects that were most successful.

7.3 QUESTIONS FOR FURTHER INVESTIGATION

Student Interest in Mathematics

In response to a questionnaire item, half of the class members stated that the calculus class did not affect their interest in mathematics and one third of the class members said that their interest increased during the pilot study. Since some of the class members had a very intense interest in mathematics when they entered the pilot study, the likelihood of increasing their interest would be very low. Hence, it seems reasonable to state the following Hypothesis: The small group-discovery method does not decrease student interest in mathematics. For many students, the small group-discovery method increases interest in mathematics.

Skill in Problem Solving

On the comparative final exam, the performance of the small group-discovery class was slightly better than that of the lecture class, but the difference was not statistically significant. Although the comparatively good performance could be attributed to the Hawthorne effect or to the ability level of the students, another interpretation is that the small group-discovery approach increases student skill in problem solving. The difference on the final exam might have been greater if the problems were harder. Moreover, more than half of the students reported that their problem solving skill increased during the pilot study. Hence, the following hypothesis seems plausible.

Hypothesis: The small group-discovery method increases student skill in problem solving in the particular subject area that is learned by the method.

An increase in problem solving skill might occur because of a practice effect, since students in the small group-discovery approach spend almost the entire class period every day solving problems. One might suspect that heavy emphasis by the teacher upon heuristic techniques would provide an additional force in the direction of increased skill in problem solving. Hence, there follows the

Hypothesis: The small group-discovery method increases student skill in problem solving in a particular subject area, and the increase is greater if the teacher places a heavy emphasis upon the use of heuristic techniques.

An increase in problem solving skill might be restricted to the particular subject matter learned by the students in a course, or the increase might generalize to other mathematical subject matter. A general increase in problem solving skill may be regarded as an increase in mathematical creativity, and it is not clear whether creativity can be encouraged by a suitable instructional program (Meyer 1969). Hence, it is appropriate to state a question, rather than a hypothesis.

Question: Can mathematical creativity be encouraged through participation in a small group-discovery approach over an extended period of time?

The Length of Class Meetings

During the pilot study, students frequently stayed and worked on problems for ten or fifteen minutes after the official end of the class period. It therefore appears that a fifty minute class period is too short for use with the small group-discovery method. This raises the possibility that mathematics classes might be scheduled for longer periods of time than fifty minutes, as are laboratories in science classes.

Question: For college students learning by the small group-discovery method, is the optimal length of time for class meetings more than one hour?

The previous question may be stated in an overly simplistic form. During the pilot study, the students were visibly more interested in some topics than in others. Moreover, the students became quite frustrated in working out the proofs of a few of the more difficult theorems, and they did not appear to be eager to stay after class and continue their work. In addition, when working on numbered lists of rather short problems, the students tended to stop work soon after the sounding of the bell--apparently planning to continue the work on the next day. It appeared that students stayed on to work after class in order to finish a problem that they regarded as interesting and challenging and which they thought they could solve. From these observations follows the

Hypothesis: The optimal length of class meetings is longest when students feel that they are making headway in solving problems that they find both interesting and challenging.

The optimal length of class meetings is shorter if one or more of the three stated conditions is not met.

Self-Pacing

During the pilot study, all the groups started the same material on the same day. There was self-pacing in each group in the sense that any group which moved very quickly was given the opportunity to solve some challenging extra problems. In addition, slower groups were not prodded to try to keep up with faster groups.

Another form of self-pacing is possible when different groups are not made to start at the same point on the same day. If different groups moved at different rates, then an introduction of new material by the teacher to the entire class would not be feasible. Instead, there should be a set of written materials containing definitions, terminology, questions for investigation, and problems. Each group could then work with the materials at its own pace, with the options of making certain changes in the topic order, deleting certain topics, and adding some questions for investigation.

Hypothesis: There is a feasible variation of the small group-discovery method in which each group works at its own pace with a suitable set of written materials.

The main difficulty with the second form of self-pacing appears to be in the formation of suitable groups. Switching from group to group appears to be rather difficult when the groups do not cover the same material at the same time. The second form of self-pacing seems to require the formation of enduring groups, composed of individuals who function well together as a social unit and who work at roughly the same rate.

Formation of Work Groups

The formation of effective and satisfying working combinations is not a simple matter. During the pilot study, three fourths of the class members tried to avoid working with various people. Different people were disliked for different reasons, and only one fourth of the class members reported never working in a group with someone they did not like. In view of these difficulties, there is a need for developing guidelines for group formation.

A few guidelines for group formation are readily apparent. In the first place, difficulties occur if the same group includes two members, both of whom have a strong desire to be the leader. Secondly, a person who wants to be a group leader should not work with people who resent a leader. A person who wants to be the leader belongs in a group with members who will tolerate or who want a leader.

Some people are bothered by the presence of a silent group member, while others are not. Some rather quiet individuals will speak up and participate actively in a group with other quiet individuals but will not participate in a group containing a very aggressive or noisy individual. In addition, some people are more concerned about interpersonal relations than about the subject matter. The presence of one such person in each work group may reduce conflict and help maintain the group as a social unit. On the other hand, some people who are concerned about interpersonal relations may not wish to work with other people who are not. Finally, certain students prefer to work with others of the same sex, whereas other students prefer members of the opposite sex or have no preference.

As far as approaches to the content are concerned, it makes sense to form groups of individuals who work at roughly the same pace. Moreover, a group may contain two individuals who consistently approach problems in different ways and who also have trouble communicating their ideas to one another. Such individuals might best work in different groups.

In summary, a tentative set of guidelines has been presented to assist the students in the process of group formation. No doubt, a number of other guidelines are necessary for group formation, and these might best be developed with the help of a psychologist. It appears that student responses to a suitable questionnaire might be used to predict the effectiveness of different work groups without going through an extended period of trial and error. Since this problem seems to be rather difficult, it is stated in the form of a

Question: Can procedures be developed to facilitate the formation of effective and satisfying work groups?

Aptitude-Treatment Interaction

One of the thorny problems for future investigation is that of aptitude-treatment interaction. What aptitudes should students have in order to thrive under a small group approach? What aptitudes should students lack in order to thrive under a small group approach? It is perhaps easier to make some educated guesses about the second question than about the first. A person who is intensely competitive, who has strong needs for recognition from the teacher, who works best under pressure, and who is most strongly motivated by grades does not seem to be well matched with the small group-discovery approach. The usual motivational sources for such a person are lacking in the method, since part of the method is the generation of a non-coercive, low-pressure environment with an emphasis upon cooperation and little emphasis upon teacher praise for individual achievements. Also not well matched with the method are individuals who have little respect for the opinions of fellow students or who have a high need to perceive the teacher as an authority figure who will prescribe and control their behavior. The set of individuals not likely to enjoy the method further includes people who desire to cover a very large amount of subject matter, who prefer a presentation by the teacher, or who like a highly structured course with a definite content syllabus.

Some amplification of the previous remarks is in order. A belief in competition would not in itself guarantee that the person would not thrive in the small group approach. For example, "Rita," who has been talked about in Chapter IV, was a "firm believer" in competition. However, she competed in a very cheerful manner and enjoyed explaining problem solutions to others. She worked very well in a group containing students who liked to have her explain things to them. Moreover, a very shy person such as "Larry" (Chapter IV) was able to enjoy class and learn in the small group approach. Hence, shyness would not be an over-riding handicap in the small group approach, provided that the shy person could find a group in which he was well accepted.

Certain aptitudes have been described which might prevent a person from achieving well and enjoying himself in a small group approach. Anyone who did not have that set of aptitudes would appear to be well matched to a small group instructional approach. However, the set of aptitudes represents speculations of the author which need to be tested by empirical investigations.

Hypothesis: A set of aptitudes can be described which reliably predicts the matching or non-matching of individuals with a small group instructional approach.

It would be desirable to develop a screening procedure to match the individual with the appropriate instructional treatment. The screening procedure might be a psychological test. Another alternative is the presentation to students of a several-page description of the small group approach. The description could include some statements about the aptitudes which indicate suitable matching or non-matching with the instructional approach. The decision to participate or not could then be left up to the individual student.

The question of aptitude-treatment interaction leaves the author with a rather uneasy feeling. Part of the reason for the uneasiness is the fact that people can change in certain ways. A person who is intensely competitive, who has strong needs for recognition from the teacher, who works best under pressure, and who is strongly motivated by grades might come to believe that his behavior has been shaped by environmental influences over a period of years. Such a person might (or might not) wish to learn some new response patterns to cope with situations unlike those he had previously encountered. The person might voluntarily choose to participate in a small group approach where the new response patterns would be accepted and fostered. Hence, the small group approach can be regarded as a possible socializing or therapeutic influence for individuals who wish to change certain aspects of their behavior and attitudes. As one girl put it, "The ability to cooperate was the most important thing I learned" (Q 88). The possible socializing or therapeutic influence of the small group method thus raises another

Question: Can a screening procedure be developed which reliably identifies those individuals with contra-indicated aptitudes who wish to change those aptitudes?

Extension of the Small Group Approach

If a small group approach has a socializing influence upon college students, it is reasonable to assume that a small group approach can also have a socializing influence upon high school, junior high, and elementary school students. There would be no need for screening procedures at the college level if people learned how to work well with each other in groups at an earlier age. As reported by R. Davis (1967), the Nuffield project has done some exciting work with small groups of students at the elementary level. These observations lead to another set of

Questions: Can a small group approach be used in mathematics instruction at all levels, from elementary school through graduate school? What would be the effects upon the learning of mathematics, the quality of interpersonal relationships, and a host of other pertinent variables?

Still maintaining an emphasis upon human relations, one may inquire about the implications of a small group approach for the classroom behavior of the teacher and thus for teacher training programs. A frequent difficulty of student teachers is the establishment of sufficient control over the class members so that learning may take place. One of the major "discipline" problems is to quiet down the class so that the teacher can be heard. This discipline problem is defined away by a small group approach, in which informal conversations between students are regarded as a virtue, not a vice.

Hypothesis: A small group approach can greatly reduce classroom "discipline" problems.

A Possible Technique to Improve Group Functioning

During the pilot study, the teacher observed that students usually did not criticize behaviors which were clearly detrimental to the progress of their work group. This observation was confirmed by the students on the questionnaire. Moreover, the attempt to promote cooperation was only partly successful and three fourths of the class members sometimes ended up working with people whom they did not like. The development of guidelines for group formation and the use of an aptitude-treatment screening procedure may have some effect in reducing these difficulties. Another approach is possible--namely the use of some modern techniques for improving group functioning and interpersonal relations.

If the diagnosis of the teacher is correct, many of the interaction problems could be resolved in an atmosphere of greater frankness and openness than occurred during the pilot study. For example, if a person attempts to dominate his group, competes in an overly aggressive manner, or refuses to listen to the views of others, there is a possibility of changing his behavior, provided that other group members are willing to tell him what he is doing and why they do not like it. This the students in the pilot study were not willing to do. Moreover, if group members are willing to frankly discuss the issue of working well or not working well with specific individuals, at least two benefits might be expected to occur. First, there is the possibility of ironing out the interaction difficulty. Secondly, there is the possibility of forming good working combinations.

The particular approach being suggested is the use of a weekend retreat for a "T-group" which is run by a qualified psychiatrist or qualified psychologist. In such a group the emphasis is upon feelings and interpersonal relations, not upon intellectual matters. The group leader attempts to create an atmosphere of warmth, trust, acceptance, and openness, in which individuals can express their feelings in contrast with their opinions. In such a group, it is often necessary to express anger or dislike before moving on to feelings of warmth, acceptance, and joy. If the group works well, ordinary task relationships are transformed as the members begin to drop their conventional masks and social roles and to become known to one another as individuals (Schutz 1967).

The weekend T-group is not being proposed as a panacea but simply as an alternative for exploration. In a method of mathematics instruction which depends heavily upon interpersonal relationships and which is intended to counter loneliness, anxiety and alienation, it seems appropriate to consider the use of psychological methods which may result in better group functioning, more meaningful interpersonal relationships, and personal growth. It is recognized, of course, that not all members of a mathematics class may like such a group or wish to participate, and there is no intention to coerce or cajole them into doing so.

There now follows the

Question: In a small group mathematics class, what are the effects of a weekend T-group upon interpersonal relationships and group functioning and individual growth?

Grading Policies

The teacher was not happy about the use of take-home examinations during the pilot study. He became quite well acquainted with most of the class members through daily conversations with them in their work groups, and it was quite hard to have to give lower grades to some of the students. The passing out of grades on a differential system was quite antithetical to the prevailing social climate in the class. It seems desirable to try an alternative grading scheme--perhaps the one used by Kingsbury (1963). In that system the students turn in homework problems to other student "correctors," and each student has the opportunity to turn in a problem as many times as necessary until he gets it right. Perhaps the use of such a grading system would promote learning and not force the teacher into the role of an evaluator and critic.

Question: Does there exist a humane grading system for use with the small group-discovery method?

Development of Curriculum Materials in Calculus

During the pilot study, the teacher intended to provide a learning environment in which the students perceived every mathematical topic as being either interesting or useful. This attempt was clearly not successful. Most of the students perceived certain topics--proofs involving the definition of the limit and the development of the natural logarithm as an integral--to be not interesting, not useful, and extremely difficult. Conceivably, these findings were unique to the particular group of students or were caused by the inherent difficulty of the small group-discovery approach. On the other hand, the findings might be generalizable to a wide population of students learning calculus by any method of instruction. The author is sufficiently convinced of the second assertion to state it as a Hypothesis: In any method of calculus instruction, the vast majority of beginning students of calculus find the following topics to be not interesting, not useful, and extremely difficult--proofs involving the definition of the limit and the development of the natural logarithm as an integral.

During the pilot study it was also found that many students perceived the examples of discontinuous functions to be artificial, contrived, and somewhat unreal. It also appeared that most of the students thought of a real-valued function as a continuous, differentiable function with finitely many relative maxima and minima. These observations lead to a hypothesis, the confirmation of which would necessitate alterations in either the beginning calculus course or in previous mathematics courses in high school. Hypothesis: Beginning students of calculus can master and feel comfortable with the topic of continuity only if they have extensive previous experience with discontinuous functions.

It is not an easy matter to provide a beginning calculus course in which students perceive every topic as being interesting or useful, and learnable. The author believes it is possible to develop curriculum materials for such a course, but to do so may require the postponement of certain topics to later courses, the provision of certain prerequisite experiences in earlier courses, and the alteration of the timing or presentation of other topics.

The production of a set of written curriculum materials for a small group approach in calculus is a primary order of business. During the pilot study, it was a time-consuming chore for the teacher to write dittoed notes every day. The imposition of such a requirement would probably be enough to convince most teachers not to use a small group approach. The written materials for the small group-discovery method would contain definitions, terminology, theorems, questions for investigation, problems, and answers to some of the problems. There would be a teacher's manual including alternate solutions to problems, appropriate hints for the students, and a listing of expected student difficulties as determined through empirical investigation.

Hypothesis: It is possible to develop curriculum materials for small group instruction in calculus so that students will perceive every topic as being interesting or useful, and learnable.

Curriculum Development via the Small Group-Discovery Method

In the pilot study students with better than average previous achievement in mathematics learned as much material as other students taught by an expository approach. This finding suggests certain hypotheses for further investigation. The hypotheses are stated in terms of mathematical ability, and they can be restated in terms of mathematical achievement.

Hypothesis A: In a given period of time, any mathematical subject matter which can be understood by average mathematics students in an expository method of instruction can be developed by bright students in a small group-discovery approach.

If this hypothesis were valid, there would be implications for both instruction and curriculum development. In terms of instruction, bright students would not have to listen to lectures but could instead develop the mathematics themselves in small groups, with no loss in time. The expository teacher could set a pace which was comfortable for average students, while the bright students were in another class developing the mathematics in small groups. Hence, there might be advantages for both sets of students. The small group-discovery approach for bright students could serve as an alternative to the placement of bright students in special sections aimed at covering more material.

In terms of curriculum development, the observations of bright students doing mathematics in a small group-discovery approach would have implications for the expository approach with average students. If Hypothesis A were empirically confirmed, its contrapositive would imply that material which could not be developed by bright students in a small group-discovery approach could not be understood by average students in an expository approach. Hence, if further investigations showed, for example, that bright freshmen could not do satisfactory proofs of limit theorems in the discovery approach, then average students could not understand those proofs in an expository approach. Thus, investigations involving small groups of bright students would have implications for expository instruction with average students. The converse of Hypothesis A is proposed as another hypothesis for investigation.

Hypothesis B: In a given period of time, any mathematical subject matter which can be developed by bright students in a small group-discovery approach can be understood by average students in an expository approach.

Hypothesis B--if it were confirmed--would also have implications for curriculum development. By observing small groups of bright students at work, it would be possible to tell what subject matter could be understood by average students in an expository setting. Hence, Hypotheses A and B, taken together, would show that curriculum development for large numbers of average students in mathematics could take place through observations of small numbers of bright students in a small group-discovery approach.

Hypotheses A and B make a distinction between bright students and average students. The following hypothesis draws a connection between the small group-discovery approach and an expository approach for students of the same ability level.

Hypothesis C: Any mathematics which can be understood by students of a given ability level via an expository approach can be developed with a slower pace by students of the same ability level in a small group-discovery approach.

The converse of Hypothesis C is

Hypothesis D: Any mathematics which can be developed by students of a given ability level in a small group-discovery approach can be understood with a somewhat faster pace by students of the same ability level via an expository approach.

Hypotheses C and D--if they were confirmed in appropriate investigations--would provide an alternate approach for curriculum development. These hypotheses raise the possibility that curriculum decisions for an expository approach could be made by observations of small groups of students in a discovery approach, keeping the ability level of the students constant. Hence, it is hypothesized that small discovery groups of students would identify all the difficulties encountered by students of a given ability level in an expository approach. However, the time taken by the discovery groups would be longer than the time for the expository approach.

In investigations pertaining to Hypotheses A-D, certain conditions should be satisfied. In the first place, it takes students some time to learn how to function in a small group-discovery approach. A fair test of the hypotheses would allow enough time for the students to become comfortable with the approach and to form an effective working group. In the second place, the hypotheses are intended to apply to units of content taking several weeks, rather than to particular topics taking only a few minutes.

The previous discussion is not intended as an endorsement of expository teaching in mathematics but as a form of recognition that expository instruction will be around for quite some time. At a later point in the chapter, a proposal is presented for a type of small group instruction which might, in the long run, serve as a replacement for expository instruction.

Investigations with One Group at a Time

For investigations of student learning of mathematics, which might or might not be intended to generate new curricula or curriculum materials, it seems to be useful to work with only one group at a time. In a small group-discovery approach operated with several groups simultaneously, the teacher may miss a substantial amount of information pertaining to the ways students go at the subject matter. While the teacher is working with one group, significant events may be transpiring in another. There follows a description of several small group investigations at the college level in which the investigator worked with only one group at a time. The studies were all conducted by graduate students at the University of Maryland. One of the studies was complete at the time of writing of this paper, and the other studies were still in progress.

A Completed Investigation

Theodore Eisenberg (1970) and Ronald McKeen (1970) completed their doctoral dissertations with a study based upon the small group viewpoint of the author and the behavioral objectives-learning hierarchy viewpoint of

Henry Walbesser (1970). The purpose of the study was to develop a procedure for generating learning hierarchies through observation of students at work in small groups. The procedure developed by Eisenberg and McKeen was a variation of the small group-discovery method in which the democratic leadership style of the instructor was replaced by a laissez-faire style of leadership. The procedure included an elaborate set of restrictions upon the permissible activities of the instructor. These restrictions were intended to guarantee that the learning hierarchies were generated by the students with a bare minimum of guidance from the instructor. During each meeting of a small group, one of the investigators served as instructor while the other investigator served as an observer and recorder.

The particular hierarchy in the Eisenberg-McKeen study had to do with the topic of limits of sequences. The first discovery group of four students was presented some broadly stated questions for investigation pertaining to sequences. In their struggle to answer the broadly stated questions, the students identified certain prerequisite tasks which had to be mastered before it was possible to resolve the original questions. Roughly speaking, the problems, questions, and skills identified by the first small group became the input for the second group of students. The process was then repeated, and the output of the second group became the input for the third group. After the third group was finished, the investigators thought that they might have a valid learning hierarchy, generated mainly by students. The investigators then presented their hierarchy on sequences to two student populations, using expository methods of instruction. The hierarchy turned out to be almost valid for students with a previous history of high academic achievement, but not valid for students with a previous history of low achievement.

Investigations in Progress

The following studies were still in progress at the time of writing of this paper. Using the previously described variation of the small group-discovery method, Donald Shriner and Neil Seidl investigated the generation of learning sequences by small groups of high achievers and by small groups of low achievers. Shriner and Seidl planned to compare the learning sequences generated by the different achievement groups and to see if the two sequences were different in any significant ways. Their doctoral study was based upon the assumption that differing achievement populations of students require different instructional treatments. Hence, it seemed possible that small groups of low achievers would generate learning sequences that could be used for instruction of other low achievers. It was also hypothesized that the groups of low achievers, in comparison with the groups of high achievers, would require more time and more teacher guidance to complete their learning sequences. The algebraic properties of certain sets of 2X2 matrices provided the subject matter for the investigation.

In the beginning stages of his doctoral project, Raymond Spaulding presented certain problems in group theory, geometry, and combinatorial mathematics to small groups of students. His intention was to develop some new content units for a course in mathematics for the elementary school teacher. Through observation of one small group at a time and through the use of attitudinal questionnaires, Spaulding determined the student reactions to the various problems and content items. On the basis of this information, he decided to throw out certain topics, revise the presentation of others, and include some new material. The trials and decisions were made before going into the classroom with a new set of materials.

For her master's thesis, Mary Poppendieck conducted a non-standard evaluation of the effectiveness of a course in mathematics for the elementary school teacher. She asked each small group of prospective elementary school teachers to design a unit for teaching arithmetic to third graders. The prospective teachers were permitted to use their textbook and class notes from the existing mathematics course but were not permitted to use any other references. The effectiveness of the course would be confirmed if the students constructed a mathematically accurate, well-motivated, and teachable unit for third graders. The effectiveness of the course for the intended population would be under serious question if the students constructed a mathematically incorrect, poorly motivated, or overly abstract unit for use with third graders.

In each of the cited studies, there were four students per group, and each group met for a period of at most four weeks. The students were grouped homogeneously according to their previous level of academic achievement, and each investigator met with at least four different groups.

The Optimal Size of a Work Group

Determination of the optimal size of a mathematics work group is a matter for further empirical investigation. There might be differences in productivity or in morale between the 3-member mathematics work group and the 4-member work group. Moreover, there were some indications during the pilot study that the group size might vary depending on the type of problem. Perhaps the 4-member group is best for problems which require the generation of many ideas (divergent thinking). Possibly, the 3-member group--or even the 2-member group--is best for problems which require fewer ideas but greater efficiency and accuracy in applying those ideas in detail (convergent thinking). Moreover, the smaller the group, the more opportunity there is for practice in problems requiring convergent thinking of a computational nature. These observations lead to the following

Questions: Is there an optimal group size for a mathematics work group? Does the optimal group size vary with the type of problem?

Maximum Size of the Class

When the students were working on difficult problems during the pilot study, the teacher was barely able, with considerable effort, to provide adequate supervision for the three work groups. Given further experience with the method, the teacher might or might not be able to add on one more work group. If one allows four members in each group, the maximum class size for efficient learning is then either twelve students or sixteen students. This observation can be restated in the form of a

Question: In a small group-discovery class in calculus, do group efficiency and individual learning both decrease as the class size increases past (a) twelve members, (b) sixteen members?

Two proposals are now presented to help resolve the problem of class size.

Class Size in Relation to the Education of Teachers

One possible solution to the problem of class size in the small group-discovery method is to have a helper for the teacher. Helpers might be recruited from the ranks of seniors or graduate students in mathematics or mathematics education. Certain seniors and graduate students would be qualified both mathematically and educationally to work with one or two small groups of calculus students. The time invested in working with the groups might be regarded as a valuable opportunity to master the subject matter of calculus and to gain experience with an unusual method of instruction. As a matter of fact, this proposal has already been made by Turner et al. (1966). An alternate approach to the problem of class size is to use a small group approach which does not require much guidance from the teacher.

A Possible Solution to the Problem of Increased Enrollments in Calculus

In Chapter I, the suggestion was made that small group instruction could be used in conjunction with either activity learning or discovery learning, depending upon the desired level of student thinking. In a small group activity approach, in contrast with the discovery approach, there would be a textbook to present the theory and the techniques for solving various classes of problems. The students would read and discuss the material in the book and would gain mastery by working problems related to the readings. The small group activity approach, in common with the discovery approach, would not depend upon the expository skill of a teacher.

It is further suggested that a single teacher could work with many more small activity groups than with discovery groups. There are two reasons for this. In the first place, many of the problems would be less difficult in the activity approach than in the discovery approach. In the second place, there would be a textbook available to present sample solutions for various classes of problems. Hence, the activity approach would be intended to function without many hints from a teacher. The written materials, plus the opportunity for small group discussion of the materials, would provide sufficient input and assistance.

During the pilot study, as reported in Chapter I, the author chose not to address the problem of providing adequate instruction for greatly increased numbers of students in the colleges and universities. That issue can now be re-opened. It is suggested that the small group-activity approach may furnish the solution to the problem of greatly increased enrollments in calculus. The case can be made as follows.

The small group-activity approach, as argued above, would not be dependent upon the expository skill, the questioning skill, or the hint-giving skill of a teacher. Hence, the teacher in a small group-activity approach might as well be a teaching assistant as a professor. As of 1970, many colleges and universities have been able to hire enough teaching assistants to handle the discussion sections in a lecture-discussion system. At many universities those discussion sections have usually enrolled less than thirty students apiece and have often enrolled less than twenty students apiece. It is suggested that a teaching assistant, making use of a small group-activity approach, could work with those discussion sections for four or five days per week, rather than only two days per week.

To push the argument one step further, it has often been asserted that teaching assistants are not competent to organize a calculus class, prepare and deliver a lecture, ask appropriate questions of the class members, and answer difficult questions themselves. In a small group-activity approach, the teaching assistant in charge of the class would not have to do any of these things, except possibly for handling difficult questions. However, a complete set of materials for a small group-activity approach would also include a teacher's manual, in which could be found a set of empirically determined student difficulties and questions in dealing with those materials. Hence, the teaching assistant could be prepared in advance for most of the difficulties that might occur in the classroom.

The small group-discovery method was designed partly to provide support for "the person in the process" and to counter the forces toward loneliness, anxiety, and alienation in the modern university. To whatever extent the small group-discovery approach is successful in achieving these goals, the small group-activity approach can also be successful. This is because the social goals of the method are achieved by the cohesiveness of the small group and by the classroom social climate, not by the level of difficulty of the mathematics problem. (However, empirical research is necessary to determine the right level of problem difficulty in order to avoid anxiety with overly hard problems and boredom with overly easy ones.) Hence, the impersonal, anonymous characteristics of the mass lecture might be countered by the use of the small group-activity method. When no lectures are presented, there is no need for mass lecture classes.

Teaching assistants in charge of a small group-activity class would have to know a little bit about applied group dynamics. The use of certain aspects of the democratic leadership style, the promotion of cooperation, and the avoidance of conformity pressure would be as important in the small group-activity approach as in the discovery approach. These matters would be discussed in the teacher's manual; in addition, a brief training period in human relations would be desirable. However, the teaching assistant would be at least as well qualified to undertake these matters as a professor, since there are few influences to make someone more person-oriented in a four or five year doctoral program in mathematics. Moreover, the teaching assistant would not be too far removed in age or background knowledge from the students; hence, it might be rather easy for the teaching assistant to establish a friendly, informal relationship with the students. Finally, it appears that a "new breed" of college teachers might emerge as the teaching assistants who used the small group-activity approach went on to become professors themselves.

The following hypothesis deals with the substitutability of the small group-activity approach for an expository instructional approach. The hypothesis is not meant to be restricted to calculus instruction.

Hypothesis: Any mathematics which can be learned by students of a given ability level via an expository approach can also be learned with no loss in time by students of the same ability level in a small group-activity approach.

A Small Beginning of a Longitudinal Study

During the one-year pilot study, there was a change in class behavior in the direction of a more relaxed atmosphere with better group functioning. Moreover, the attitudes of the students toward the class either stayed the same or changed in a positive direction. In addition, a number of students requested the continuation of the small group-discovery class for another semester. These observations taken together suggest the desirability of a longitudinal study.

Question: If mathematics majors took four years of mathematics courses via the small group-discovery method, would there be positive effects upon their mathematical knowledge, skill in problem solving, interest in mathematics, success in graduate school, skill in interpersonal relations, capacity for empathy, and other pertinent variables?

A substantial amount of preliminary work needs to be undertaken before one thinks seriously about any longitudinal study. The pilot study provided strong support for the contention that calculus can be taught via the small group-discovery approach, given a small class of students with better than average previous achievement in mathematics courses. The work of R. L. Moore and some of his students in topology has demonstrated that some people are able to learn topology by an individual discovery approach. The author believes that any student who can learn topology by the Moore method can also learn topology by the small group-discovery method, provided that he enjoys working with other people. This assertion needs to be tested by an empirical investigation; however, rather than beginning with a subject where there is a high probability of success, it seems appropriate to undertake preliminary work in an area not previously considered suitable for a discovery learning approach. In the fall of 1970 an attempt will be made to teach an undergraduate abstract algebra course by the small group-discovery method. Before embarking on this venture, it is best to provide a summary of the present work.

7.4 SUMMARY OF THE PAPER

This paper reports the development of a new strategy of mathematics instruction in which students work together in small groups to discover mathematical results. The strategy is named the "small group-discovery method." To determine the feasibility of the method, a one-year pilot study was conducted with a class of college freshmen studying calculus.

The strategy was designed largely in accordance with the educational philosophy of John Dewey. The students learned mathematics by doing mathematics, and the approach to the subject matter was one of guided discovery. The students, with some guidance from the teacher, stated and proved the theorems of calculus and developed techniques for solving various classes of problems. The classroom activity was a social process taking place in small groups of students, with four members in each group.

The teacher spent most of the class period with the small work groups--giving hints, checking solutions, providing encouragement, and so forth. The teacher provided mathematical input in the form of questions for investigation with appropriate terminology; this was done in brief discussions with the entire class, generally lasting for no more than five or ten minutes per day. There was no textbook for the course; the textbook was replaced by a set of dittoed notes prepared by the teacher.

Interest in the mathematics was intended to provide the major source of motivation, and there was a non-threatening atmosphere in which students could feel free to express their opinions. Emphasis upon grades was substantially reduced by an A-B grading scale and there were no examinations in class. The teacher attempted to determine which topics had intrinsic value, which appeared to be useful, and which had no value from the student viewpoint. Professional standards of rigor were not imposed upon the beginner, and the presentation of subject matter proceeded from the more concrete to the more abstract, as seen by the learner. Skills were formed under conditions requiring thought, and there were opportunities for the exploration of student-generated questions.

Decisions about the classroom social climate were made on the basis of experimental studies in group dynamics. The teacher made use of a carefully specified style of democratic leadership, rather than an autocratic or laissez-faire style. Within each work group there was to be an atmosphere of cooperation, rather than competition. Conformity pressure was reduced through the development of standards which encouraged members to conform to their own judgment, and the work groups operated without designated leaders. The teacher exerted his influence to limit the discrepancy in power between the most active and the least active group members, and efforts were made to facilitate the participation of all members in the discussion. The group goal was not only to solve the problem, but to solve the problem in such a way that all members understood the solution. The total class consisted of twelve members, who worked in three small groups having four members apiece.

The students in the one-year pilot class were all volunteers who were at least mildly interested in mathematics and who had A or B grades in their high school mathematics courses. On a comparative final examination, the students in the pilot class scored slightly better, but not significantly better, than the students in a control section taught by the lecture-discussion system. However, some students in the control section would not have satisfied the entrance requirements for the discovery class.

The students in the discovery class performed very credibly on seven take-home examinations. An open-ended questionnaire showed that the pilot class had either positive or non-negative effects upon each student's interest in mathematics and estimate of his problem solving skill. There were numerous indications that the small group-discovery approach deserves intensive investigation as an approach for both instruction and curriculum development. A number of problems were stated for future investigation with the small group-discovery method.

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Appendices A, B, C, and D have been deleted from this report, but are available on microfilm at the University of Wisconsin Memorial Library.

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